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Fuzzy product -limit estimators: Soft computing in the presence of very small and highly censored data sets

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FUZZY PRODUCT-LIMIT ESTIMATORS: SOFT COMPUTING IN THE PRESENCE OF VERY SMALL AND HIGHLY CENSORED DATA SETS

by

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We hereby recommend that the dissertation prepared under our supervision by Kian L. Pokorny entitled Fuzzy Product-Limit Estimators: Soft Computing in the Presence of Very Small and Highly Censored Data Sets be accepted in partial fulfillment of the requirements for the Degree of Ph.D. in Computational Analysis and Modeling.

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ABSTRACT

When very few data are available and a high proportion of the data is censored, accurate estimates of reliability are problematic. Standard statistical methods require a more complete data set, and with any fewer data, expert knowledge or heuristic methods are required. In the current research a computational system is developed that obtains a survival curve, point estimate, and confidence interval about the point estimate.

The system uses numerical methods to define fuzzy membership functions about each data point that quantify uncertainty due to censoring. The "fuzzy" data are then used to estimate a survival curve, and the mean survival time is calculated from the curve. This estimator converges to the Product-Limit estimator (Kaplan and Meier, 1958) when a complete data set is available. Several measures of uncertainty are considered. Using a modification of the Bootstrap method (Efron, 1979) an estimate of both the random uncertainty from the data, and the vague uncertainty from the censoring in the data are calculated. In addition, this method allows for the incorporation of expert knowledge into the estimator and the weighting of data groups. In the latter situation, a group of units that has had much time in service is given larger weights and a group of units that have had little time in service is given smaller weights when making the estimates.

The estimator is tested under several circumstances. Data are generated from several distributions, the estimates are made, and the results are compared to the distribution parameter and estimates made using the Product-Limit Estimator.

DEDICATION

TO THE MEMORY OF BRUTUS R. POKORNY, A LOYAL FRIEND

TO MY WIFE KIMBERLY

TO MY DAUGHTER ALEXANDRIA

WITH LOVE

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CHAPTER 1

INTRODUCTION

1.1 Introduction and Research Motivation

Maintenance and repairs represent significant costs for organizations that operate machinery. To minimize these costs, the time of failure should be predicted with accuracy. This prediction of failure or estimate of reliability must be accurate to predict cost and operate efficiently. In some environments, developing accurate prediction of failure or estimates of reliability is problematic. It is upon these types of situations that this thesis is focused. For example, failure or reliability is typically evaluated based on significant quantities of historical data, but what can be done when new technology is employed where there are very limited historical data?

As an example of this type of problem, the reader is asked to consider the use of a new technology where there are no historical data. In this case, a piece of equipment with associated reliability estimates such as Mean Time To Failure (MTTF) and Mean Time Between Failures (MTBF) are obtained from laboratory tests. The equipment is then put into use in the field. The current (field) "operating time" of the equipment should provide information for updating the reliability estimates before any failures.

More generally, reliability estimates on a piece of equipment under one mode of operation are available, and it is desirable to use this information along with the operation time under a new mode of operation to obtain point estimates and confidence bounds for the new mode of operation.

This problem can be reduced to a problem of right censored data because the operating time of the "in service" equipment is used to make estimates before the equipment failing. The Product-Limit estimator (Kaplan and Meier, 1958) provides a way to estimate a survival curve with right censored data. The mean of this curve gives a point estimate of the mean time of survival. A caveat in the current situation is that the test data and the field data may be from different distributions. Another problem is that there are very few data, and some of the test data appear to contain errors (outliers). In addition, the Product-Limit estimator gives erroneous results under the conditions that will be considered in Chapter 2.

Manufacturers can be reluctant to provide raw test data to the purchasers of their equipment, and testing may be expensive. Also, the manufacturer's testing objective and the purchaser's objective differ. Units costing hundreds of thousands of dollars are put into service. Often very few units (one to ten) are purchased and used. Therefore, the use of the information gained by putting the unit(s) to use in the field becomes dependent on the amount of data provided by the manufacturer.

The old mode of operation may contain fewer than five data points, and the new mode may also consist of a data set of size fewer than five. The idea is to extract as much information as possible from the data. In addition, it is necessary to quantify and gain an understanding of the error or uncertainty involved in the estimates made from this limited data set. Some of the uncertainty is in the form of statistical error while some is in the form of vague uncertainty about the data. The latter is due to the censoring because at the time of censoring it is known only how long the unit survived, not when it failed. In this research both of these uncertainties are used with the bootstrap method to provide confidence bounds for the estimate.

Statistical methods are dependent on the type and amount of data. With large sample theory, many methods based on asymptotic properties of the estimator are available. These methods appear in many standard texts (Hogg and Craig, 1978; Mason, Gunst and Hess,1989; Casella and Berger, 1990). These methods become unreliable, producing erroneous results and answers with significant uncertainty, as the sample size becomes smaller and assumptions about the data are violated. Non-parametric methods reported by Hollander and Wolfe (1973) are available in some situations when assumptions about underlying properties of the data cannot be made. Resampling methods like bootstrapping (Efron, 1979) are available with small data sets. The bootstrap method is normally not recommended for sample sizes of fewer than ten because it too is based on certain asymptotic properties. In general, for data sets of size fewer than ten, not much can be done in the way of statistical significance. When fewer data are available, expert knowledge tends to be a better approach.

In general, uncertainty increases as the amount of available data decreases and the variability in the data increases. Consider two data sets $\boldsymbol{x}$ and $\boldsymbol{y}$ each of size n = 5. If $\boldsymbol{x} = \{1, 1, 1, 1, 1\}$ and $\boldsymbol{y} = \{1, 12, 95, 208, 1059\}$ then the variability in the data plays a more important role than the size of the data sets. What can be said about the average value of $\boldsymbol{x}$ and $\boldsymbol{y}$? Whereas, if the size of the data set $\boldsymbol{y}$ increases without bound, it tends to the shape of the underlying distribution, and much can be said about the sample data. Here it is desired to quantify the amount of uncertainty.

What is uncertainty? Probability theory assumes uncertainty to follow a random process and the realizations of the outcomes of a random process are based purely on chance. It is not possible to predict a sequence of events. However, it is possible to give a precise description of the statistics of the long-run averages of the process. Uncertainty also comes in the form of imprecision and vagueness. Imprecise information can be associated with both quantitative and qualitative data. An example of the latter occurs with the statement that the system costs "very little" to run. An example of quantitative imprecision is to say the system costs $500 in fuel and $100 in maintenance a year to run, where it is apparent from the zeros in the cost that the answers lack exactness. If exact measurements were taken on the cost of running the equipment in ten different years, it is likely that ten different values would be realized. Vagueness is uncertainty usually associated with linguistic and intuitive information. Examples of vague information are that the engine is in "good shape," the electrical cable "felt hot," or the crewmember is "dependable." Fuzzy set theory provides a way to quantify uncertainty due to vagueness and imprecision.

In the situation under study, few data are available. Given enough data, large sample theory could be invoked, and the vast world of statistical theory is available. With any fewer data, only heuristic methods are available. In the situation under consideration however, there are enough data to provide some information. Some of the data may be uncensored and some censored. It is known that data sets of this type follow certain statistical properties. Thus, fuzzy methods are used in conjunction with standard statistical methods and resampling methods to obtain a point estimate with confidence bounds. The new methods developed in this thesis are intended to use fuzzy information when few data are available and converge to the Product-Limit estimate of the survival curve (Kaplan and Meier, 1958) as more data become available.

Consider a right-censored data set of survival times of size four, {(2,1), (4,1), (17,1), (45,0)}. Each ordered pair (t_i , d_i) indicates time (t_i) and an indicator (d_i) of whether the i^{th} time is censored (d_i=1) or not censored (d_i=0). The first three are censored; thus, all that is known is that they have survived until their given times of censoring. The last time is a failure time. It is known that this last unit failed at time 45. Assuming that the units from which the times were taken are all of the same type, then it is expected that the survival time of the first units should have some correlation with the last units. This influence is the central hypothesis of this study. With each censored data point, its failure time is fuzzy in the sense that it is known how long the unit has survived, and the survival (failure) times of the other units give indications of how much longer the unit may survive. Several numerical methods are developed to describe this influence in the form of fuzzy membership functions. A variation on the Product Limit estimator is developed to incorporate this fuzzy data. Finally, the bootstrap method is modified to account not only for the statistical error but also the uncertainty due to the censoring in the data. Measures of the amount of "vague" uncertainty quantified by the fuzzy membership functions is given in several different forms. The whole system results in a computationally intelligent system that extracts information from a small data set by allowing each data point to use the information from the data around it as an indicator of how long it may survive. As more complete data become available, the system tends to the standard statistical methods.

1.2 Organization of the Dissertation

The work in this dissertation comes from several different broad areas in mathematics, statistics, and computer science. Figure 1.1 shows the relationship of these areas and some of the sub-topics investigated to achieve the final system.

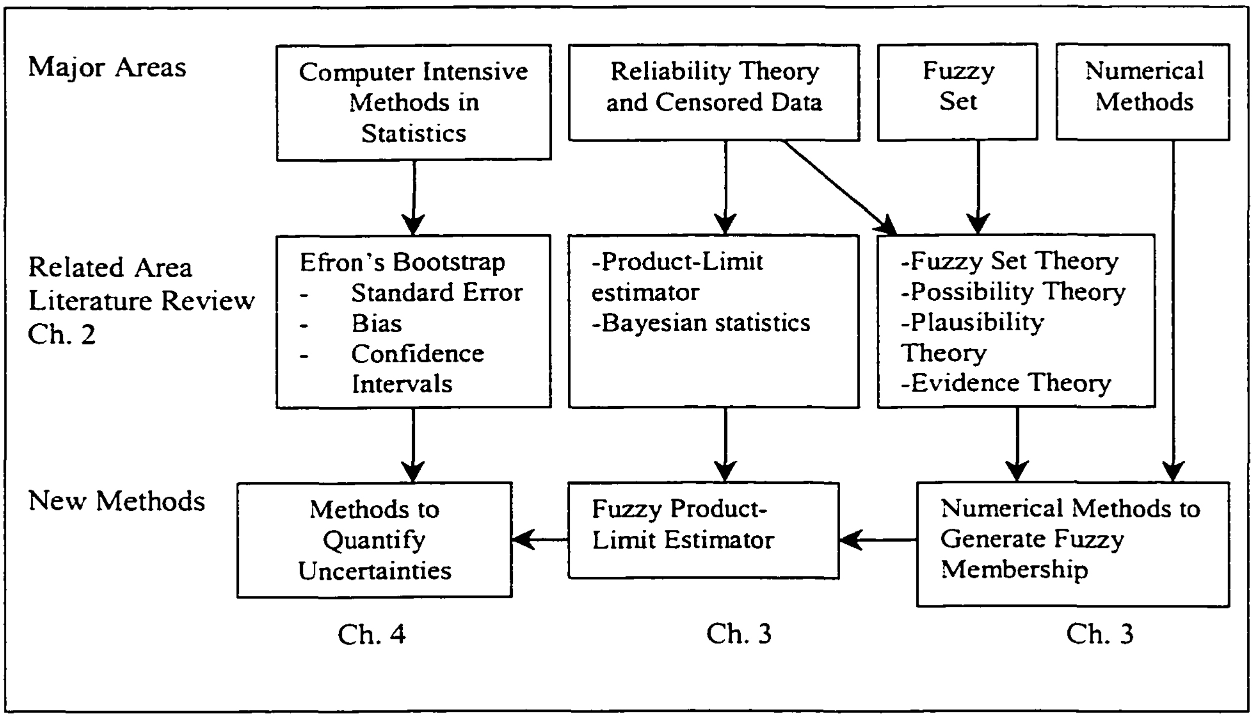


Figure 1.1 Relationship of Research Areas

Chapter 2 provides a literature review and background into several areas of research in reliability theory, computer intensive methods in statistics, and computational intelligence. Section 2.1 considers reliability theory with censored data and introduces the Product-Limit estimator. Then, section 2.2 provides an introduction to fuzzy theory. Section 2.3 introduces Bayesian and fuzzy Bayesian methods. These methods are very close in nature to the new method developed in Chapter 3 of this dissertation and are therefore included. There are, however, very fundamental and obvious differences in this new method and existing methods that are investigated. Finally, section 2.4 provides an overview of the basics of computer intensive methods in statistics known as resampling methods including the Bootstrap Methods (Efron, 1979).

Next, the new method is developed which includes the development of a numerical method to generated fuzzy membership functions about each censored data point, based on the known data. Several variations of generating membership functions from the data are developed in sections 3.2 and 3.3. In section 3.4 a new method called the "Fuzzy Product-Limit Estimator" that incorporates fuzzy information into the Product-Limit estimate is derived. To conclude this chapter, in section 3.5 statistical and asymptotic properties of this new estimator are considered.

In Chapter 4, several methods are investigated to quantify the uncertainty in the estimate. The bootstrap method (Efron, 1979) is used to obtain confidence intervals for the estimates. A method is developed to consider the uncertainty due to the vagueness of the censored data and the uncertainty due to randomness in the data associated with the estimate. Both types of uncertainty are then used to create a two-dimensional confidence interval that describes the statistical uncertainty in the abscissa direction and the vague uncertainty in the ordinate direction of the Cartesian coordinate system.

Chapter 5 provides simulations and empirical results to show how these new methods perform. This chapter tests the performance of the Fuzzy Product-Limit Estimator. Testing the estimator itself is accomplished by generating random data from several distributions consisting of censored and failure times. Next, estimates are made using the Fuzzy-PLE with associated uncertainty measures. The tests are run repeatedly for each distribution, the data analyzed, and comparisons are made with the actual mean of the distributions and the performance of the PLE on each data set.

In Chapter 6 a simulation is developed to model the situation that has motivated this research. This simulation consists of simulating the failure times of several large pieces of equipment over time. Units are put into service, failures are recorded, and at given times during the simulation estimates are made of the mean survival time. Several issues are addressed with this simulation. It is demonstrated that for early times the estimate of the mean survival time obtained using the Fuzzy-PLE is superior to the estimate obtained using the PLE, and that over time, with matured data, the estimate obtained using the Fuzzy-PLE agrees with that of the PLE.

Finally, Chapter 7 provides conclusions, and recommendations for future research with these methods.

CHAPTER 2

BACKGROUND AND LITERATURE REVIEW

This chapter provides a discussion of existing theories and methods as it relates to the proposed work. The three main veins discussed are reliability theory and censored data, fuzzy set theory, and re-sampling methods in statistics. These areas provide the foundation on which the new methods stand.

2.1 Reliability Theory and Censored Data

As it pertains to technical systems, reliability theory has roots back as far as the industrial age. Just after World War I, the concept of reliability was used in comparing the operational safety of one-, two- and four-engine airplanes. The reliability was measured as the number of accidents per hour of flight time. Walter Shewhart, Harold Dodge, and Harry G. Pomig laid down the theoretical basis for using statistical methods in quality control of industrial products in the early 1930's. These methods were brought into use with the on set of World War II. In Germany, mathematician Robert Lusser derived the product probability law of series components while analyzing the V-1 missile, a system made up of many components. In the years following World War II more complex products were being developed, and reliability became a more important issue (Villemeur, 1992).

During this era an ongoing concern existed with regards to incomplete or censored data. If a test is allowed to run until all units have failed and the life times are recorded, the data set obtained is said to be complete. Otherwise the data set is called incomplete. It may be impractical or too expensive to wait until all units have failed. In addition, data can be lost or starting times may be unknown. In this situation the data set is said to be censored (Hoyland and Rausand, 1994).

Presently, research in the area of censored data continues. Ebrahimi (1992) considers prediction intervals under hybrid censoring. Chiou (1997) investigates censored data from two different pretests and combines the data sets to make inferences. Hazard Plotting of multiply-censored data as a data analysis tool is considered by Nelson (2000). Li (1999) considers estimating the probability that a component is operating at a specified time when both the operating time and repair time are subject to right censorship. In this section, some standard statistical theories for working with censored data are considered.

2.1.1 Exponential Distribution and Censored Data

The general form for an exponential distribution for the time to failure is given by $f(x \mid \gamma, \lambda) = \lambda \exp[-\lambda(x-\gamma)]$, $0 \le \gamma \le x; \lambda > 0$. The exponential distribution can be chosen as a failure distribution if and only if the assumption of a constant hazard rate (failure rate) can be justified. This assumption implies that the failure is due to random shocks that occur according to postulates of the Poisson process. The assumption of exponentially distributed lifetime implies that a unit is stochastically as good as new, so there is no reason to replace a functioning unit. In addition, this assumption implies that for the estimation of the reliability function, the mean time before failure, mean time between failures, and mean survival time, it is sufficient to collect data on the number of hours of observed time in operation and the number of failures. The age of the units is of no interest in this connection (Hoyland and Rausand, 1994).

It is known that given an exponential distribution, the mean time to failure (MTTF) is $\gamma + 1/\lambda$ and the failure rate is $z(t) = \lambda$. An exponential distribution can be used in certain situations in which hazard rate is not constant if an appropriate transformation of the failure time data is made. If $t_{(1)},\ldots,t_{(n)}$ is Weibull(β , γ , δ) then:

$$f_T(t) = \left\{\frac{\beta}{\delta}\left(\frac{t-\gamma}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{t-\gamma}{\delta}\right)^{\beta}\right]\right., \quad t \geq \gamma; \quad \beta > 0; \quad \gamma \geq 0 \qquad (1)$$

with $x_{(i)} = (t_{(i)} - \gamma)^{\beta}$ is exponential(δ^{β}) (Mann, Schafer, and Singpurwalla, 1974).

With right censored (or Type I censoring) the life test is terminated at a specified time t_0. All units are activated at time t = 0 and followed until failure or until time t_0 when the experiment is terminated. The information in the data set is then $f \leq n$ observed ordered lifetimes ($T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(f)}$) and *n-f* components that have survived to time t_0 where *f* is the number of failures and *n* is the sample size. The number of components that fail before t_0 is stochastic. It may be that no units fail during this time (*f=0*)! In general the censoring time t_i may be different for each of the censored units.

In Type II censoring the life test is terminated at the r^{th} failure, $0 < r < n$. The information in this data set is ($T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(r)}$) and *n-r* components that have survived the time $T_{(r)}$. Here $T_{(r)}$ is stochastic. Type III Censoring is a combination of Type I and Type II. The life test terminates when the first of either t_0 or the r^{th} failure occurs. In Type IV Censoring *n* numbered identical units are activated at different given points in time. The time for censoring of unit *i* (S_i, $i = 1, 2, \ldots, n$) is stochastic (Hoyland

and Rausand, 1994). Research has been done on estimating the mean time to failure with unknown censoring (Shanmugam and Richards, 1989).

In the situation under study, it can be considered that the current use of the equipment is type I or right-censored data. Since the equipment has not failed, the right side of the time line is uncertain, in which the time axis is considered to begin to the left and extend to the right as time increases. The idea of left censored or left-truncated data may arise in the current situation as well. With left-truncated data the time line on the left or at the beginning is cut off. In other words, the time at which the unit is put into service is unknown. During the test period, however, everything is known about the unit other than its true running time. Although left truncated data are not considered directly, the method developed in this dissertation can easily be modified to account for left-truncated data.

2.1.2 Maximum Likelihood Estimators with Censored Data

The method of maximum likelihood is viewed as a means for selecting the best fitting distribution from a class of admissible distributions. Interest here is in the survival time or mean time to failure. When testing without replacement and γ is known or 0 (if it is known a transformation can be made to make it 0) then the maximum likelihood estimator of λ for Type I censoring at time t_0 is

$$\hat{\lambda} = \frac{f}{\sum_{i=1}^{f} T_{(i)} + (n-f)t_0} \tag{2}$$

This estimate is biased for small samples but behaves asymptotically (Hoyland and Rausand, 1994; Mann, Schafer, and Singpurwalla, 1974). Nachlas and Kumar

(1993) show that for doubly censored (left-truncated and right-censored) data using several heuristic methods, the estimator for λ in equation 2 is still superior. This result can be generalized such that the censoring time may be different for individual units. In the general case t_i is the censoring time for the i^{th} unit. In this case the MLE for λ is

$$\hat{\lambda}_1 = \frac{f}{\sum_{i=1}^{n}\{a_i T_{(i)} + (1-a_i)t_i\}} \tag{3}$$

where $a_i = 1$ if $T_{(i)} \le t_i$ and $a_i = 0$ if $T_{(i)} > t_i$ (Bartholomew, 1963). The MLE of λ for Type II censoring is

$$\hat{\lambda}_2 = \frac{r}{\sum_{i=1}^{r} T_{(i)} + (n-r)T_{(r)}} \tag{4}$$

(Mann, Schafer, and Singpurwalla, 1974; Gross and Clark, 1975). In either case, the number of failures is divided by the total time on test. Thus, it is sufficient to record the total time on test and the number of failures.

Now let $\theta = \lambda^{-1}$. For Type II censored data on θ, a $1-\alpha$ confidence interval is given by $\left[2r\hat{\theta}/\chi^2_{1-\alpha/2}(2r), 2r\hat{\theta}/\chi^2_{\alpha/2}(2r)\right]$ where $\chi^2_\gamma(k)$ is the 100γth percentile of a chi-square distribution with k degrees of freedom (Mann, Schafer, and Singpurwalla, 1974). A confidence interval for θ with Type I censored data is more complicated. Bartholomew (1963) shows that if the censoring time for each component is equal, the MLE for θ when the times to failure are ignored is:

$$\hat{\theta} = \frac{t_0}{-\ln(1 - f/n)} \tag{5}$$

and confidence bounds based on this are extremely efficient. Mann, Schafer, and

Singpurwalla (1974) provide a development of a 1-α lower confidence bound for θ based on Bartholomew (1963) as $t_0 / \ln(1/\rho)$, where ρ is found by solving

$$\alpha = \sum_{i=n-s}^{n} \binom{n}{i} \rho^i (1-\rho)^{n-i} = \int_0^{\rho} \frac{\Gamma(n+1)}{\Gamma(n-s)\Gamma(s+1)} x^{n-s-1}(1-x)^s \, dx \tag{6}$$

Thus, table values from the binomial, beta, or F distribution may be used. Unfortunately, equipment in use in the field is being studied in this situation, and the censoring time is not equal for each component. Thus, this statistic cannot be used for the circumstances of immediate interest.

When testing without replacement and γ is not known, with Type I censoring the MLE estimate of γ is $\hat{\gamma} = T_{(1)}$ (the time of the first failure) and the MLE of λ is

$$\dot{\lambda} = \frac{s}{\sum_{i=1}^{s} T_{(i)} + (n-s)t_0 - nT_{(1)}} \tag{7}$$

Here there are no confidence intervals for γ or $\hat{\lambda}$ (Mann, Schafer, and Singpurwalla, 1974) and the first order statistic must be known.

It has been seen here that the exponential distribution has a maximum likelihood estimator for the mean under both type I and type II censored data. Also under certain conditions there exists some confidence bounds for these estimators. These estimators are biased for small samples. In the following methods developed these estimators are considered as a guide for the fuzzy sets. The methods developed in this dissertation are non-parametric, making no assumption of the underlying distribution. Thus, data from the Weibull, lognormal, Gaussian or any other common failure distributions can be used.

2.1.3 Kaplan-Meier Survival Curve

Another method for survival data is to consider estimates of the survival curve. The product-limit (PL) estimate is a non-parametric procedure to estimate the proportion P(t) of items whose lifetimes would exceed time t, if they had not been censored, without making any assumption about the form of the function P(t) (Kaplan and Meier; 1958). This estimator can be shown to be a maximum likelihood estimator (Kaplan and Meier, 1958; Miller, Gong and Munoz, 1981). In addition, it is a consistent statistic (Kaplan and Meier, 1958; Miller, Gong and Munoz, 1981). This is the most commonly used estimator for survival curves. It has been used to develop confidence intervals for the mean residual time with censored data (Li, 1997).

To illustrate this method, consider the example directly from Kaplan and Meier (1958). A random sample of 100 items is put on test at the beginning of 1955; during the year 70 items die and 30 survive. At the end of the year, a larger sample is available, and 1000 additional items are put on test. During 1956, 15 from the first sample and 750 from the second sample die, leaving 15 and 250 survivors, respectively. As of the end of 1956, it is desired to estimate the proportion P(2) of items in the population surviving for two years or more. This particular example is such that it is easy to form an estimate P*(2) = 15/100 = 0.15 from the first sample alone. This is called the reduced sample estimate because it ignores the 1000 items tested only during 1956. It is a legitimate estimate only when the reduced sample is itself a random sample; this is the case only when the observation limits (censoring times) are known for all items. In the absence of this information, it would be impossible to discriminate among the 835 deaths observed before the age of two years. The 250 censored items of the 1000 existing at age one cannot be ignored because only 15 of the original 100 sample have survived for two years; P(2) would then be estimated as 15/850 = 0.018 an absurd result.

Now consider the estimate for one year P(1) from the two samples: by combining the samples, the estimate $P^*(1) = (30 + 250)/(100+1000) = 0.255$ is obtained for P(1). This result extracts all possible information from the second sample for the present purpose. How does it help to estimate P(2)?

The answer is that there are advantages to using the first sample for estimating P(2)/P(1), the conditional probability of survival for two years given the survival for one year, rather than P(2) itself. This estimate is $P^*(2)/P^*(1) = 15/30 = 0.5$ hence $P^*(2) = 0.255 * 0.50 = 0.127$. This is a very simple example of the product limit estimate. The advantage is that there is no requirement to know that the 750 deaths in the second sample had observation limits of one year because these items are irrelevant to estimating P(2)/P(1).

Several assumptions need to be made about the data. Some small violations of the assumptions have little effect in most situations, but others may render the results useless. In particular, small sample sizes may increase the effect of assumption violations and small sample sizes are the target data sets in the current research. Assumptions that should be considered are as follows:

(1) Independence within the sample: The failures are not correlated within the sample, in the sense that one failure does not cause or contribute to another unit failing.

(2) Independence of censoring: A pattern should not be seen in the censoring of the units. This is not an issue in the situation under study. The data are censored based on their start time and the time of recording the data.

(3) Uniform failure rate within each time interval: Because no assumptions are made about uniform failure rate within each time interval it may be of concern in the situation under consideration, but the methods under development do not require this assumption.

(4) Low percentage of censored values: Again, a low percentage of censored values may be of concern in the situation under consideration, since the target data set has high percentage of censoring. The new method is meant to improve on the Product-Limit estimator under these conditions.

Let $T_1 \le T_2 \le \ldots \le T_k$ represent the distinct failure times. Let n_i be the number of surviving units just prior to T_i , d_i be the number of units that fail at T_i, and s_i=n_i-d_i for i = 1, ... , k. The Product-Limit estimates are given by. (Kaplan and Meier; 1958):

$$\hat{s}(T_i) = \prod_{j=1}^{i}\left(1 - \frac{d_j}{n_j}\right) \tag{8}$$

That is, the ratio of surviving units just after a failure to surviving units just before the failure is calculated. Notice that this estimator is defined to be right continuous (i.e. the events at t_i are included in the estimate of $s(T_i)$). The standard error of this estimate can be calculated using Greenwood's formula:

$$\hat{\sigma}\left(\hat{s}(T_i)\right) = \hat{s}(t_i)\sqrt{\sum_{j=1}^{i} \frac{d_j}{n_j s_j}} \tag{9}$$

(Kalbfleisch and Prentice, 1980). Confidence intervals based on the sign test can be obtained for the quartiles.

Now the mean survival time is calculated by

$$\hat{\mu} = \sum_{i=1}^{k} \hat{s}(T_{i-1})(T_i - T_{i-1}) \tag{10}$$

where T_0 is defined to be zero. If the last observation is censored, this sum underestimates the mean. The standard error of $\hat{\mu}$ is estimated by

$$\hat{\sigma}(\hat{\mu}) = \sqrt{\frac{m}{m-1} \sum_{i=1}^{k-1} \left\{ \frac{A_i^2}{n_i(n_i - d_i)} \right\}} \tag{11}$$

where

$$A_i = \sum_{j=i}^{k-1} \hat{s}(T_j)(T_{j+1} - T_j) \tag{12}$$

and $m = \sum_{j=1}^{k} d_j$ (SAS, 2000). Here the mean time that a unit survives is being estimated based on the proportion of units that had survived to time T_i. Thus, the censoring time may be different for each unit.

2.2 Fuzzy Theory

In the current study, fuzzy membership functions are used to describe the uncertainty in the failure times of censored units. This technique involves an understanding of fuzzy set theory and how expert knowledge can be described by fuzzy sets. The "expert knowledge" here is in the form of intuitive information from the data set. What follows is a brief history and a description of some aspects of fuzzy set theory.

In 1965, Dr. Lofti Zadeh (1965) coined the term fuzzy sets in his paper of the same name. Although the concept was not completely new, it was, however, a more complete development of the theory of multivalued logic than had been considered previously. As early as 1917, Bertrand Russell used the term vagueness to describe the concept of multivalued logic (Russell,1917). In 1920, Jan Lukasiewicz published a paper that detailed the principles of multivalued logic. The idea is that statements could take on fractional truth-values between the zero and one of classical (binary) logic. In 1937, Max Black defined the first fuzzy set with what is now known as a membership function (Black, 1937). Black viewed everything as having a degree of membership to a set. He used the term "vague" following Russell. What Black had done was put the vagueness in symbols at the level of sets or systems. He extended multivalued logic to sets and showed that these sets matched our intuitive ideas. However, Black's theory did not become popular. It took 30 years and a well respected, tenured professor at University of California at Berkeley, using a new term "fuzzy" to get these ideas started again.

Since Dr. Zadeh's 1965 paper a great amount of work has been done in the areas of fuzzy set theory, fuzzy logic, and fuzzy control. These concepts have been applied in numerous disciplines. Mitsubishi manufactures fuzzy air conditioners that control temperature according to human comfort indexes. In videography, Fisher, Sanyo, and others make camcorders that use fuzzy logic for focusing and image stabilization. The Japanese city of Sendai has a sixteen-station subway controlled by a fuzzy computer. This controller makes seventy percent fewer judgement errors in acceleration and braking than human operators. These are just a few examples of how extensive the use of fuzzy logic is in today's technology. Fuzzy set theory has also been combined with neural networks, evolutionary computing, and other areas from artificial intelligence to form what is called soft-computing (Jang, Sun, and Mizutani, 1997).

2.2.1 Relationships Among Several Theories

There exists a relationship among fuzzy set theory, fuzzy measure theory, probability theory, and evidence theory. These theories all model and characterize various forms of uncertainty. As more information becomes available, the mathematical description of uncertainty can easily transform from one theory to the next.

In classical, two-valued logic, elements are either true or false, in the set or not in the set. This binary membership can be stated mathematically with an indicator function, $X_A(x) = \{1 \text{ if } x \in A \text{ and } 0 \text{ if } x \notin A\}$. Fuzzy sets extended the idea to include various "degrees of membership" on the continuous real interval [0, 1]. The membership function can be written as $X_A(x) = [0, 1]$. With fuzzy sets an element is in the set to some degree. Therefore, classical or crisp set theory is the degenerative case of fuzzy set theory. Classical or crisp set theory as a special case considers only the end points of the interval. "Math as we know it is but a special case of fuzzy math, a special limiting case –the degenerative case of black and white extremes in a world of grays" (Kosko, 1993).

Fuzzy measure describes the vagueness or imprecision in the assignment of an element to two or more crisp sets (Ross, 1995). A fuzzy set is a single set with uncertain boundaries. With fuzzy measures the uncertainty is in the assignment to the set and is not random. It is uncertain because the evidence to establish the assignment is "weak." If the evidence is complete then it reduces to a probability assignment. On the other hand, if the evidence is completely lacking then it is the case of total ignorance. With fuzzy sets the imprecision is in the boundaries of the set. With fuzzy measures the imprecision is in the assignment of an element to two or more crisp sets.

There are two special forms of fuzzy measure. A belief measure is associated with preconceived notions and expresses the degree of support or evidence. If the belief is 100% then there is certainty. Based on this concept ignorance can be defined as the difference between the belief in A and $\sim A$ (not A) (i.e. Ignorance = 1- [belief(A) + belief($\sim A$)]). A plausibility measure is associated with information that is possible. If the plausibility of an event is 100% then it is a possibility measure. The plausibility measure of collection A is defined as the complement of the belief of the complement of A or Plausibility = 1- belief(~A). Figure 2.2.1 shows the relationship between these theories. The properties of continuity, associativity, distributivity, idempotency, identities, transitivity, and involution all hold for fuzzy sets.

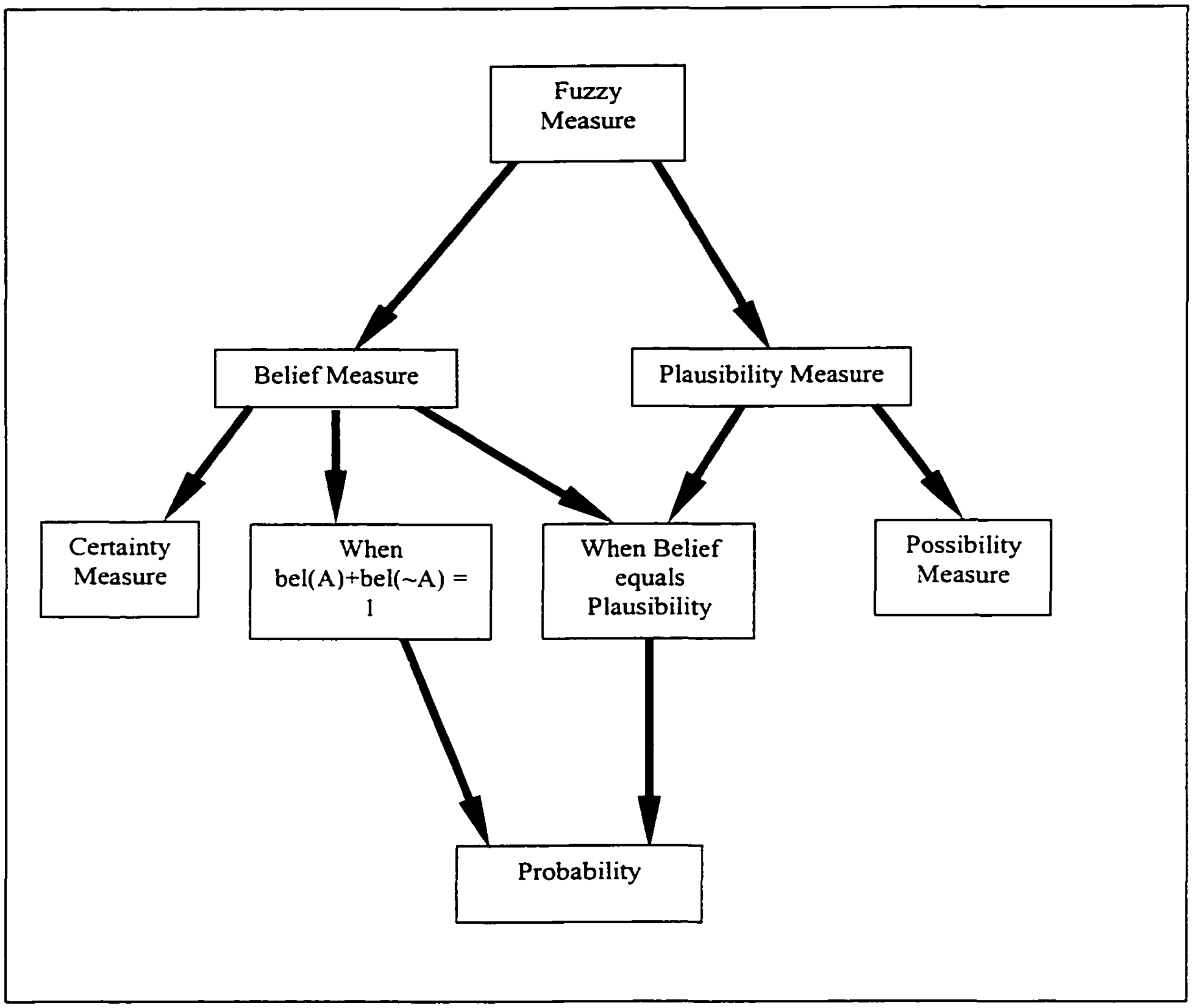


Figure 2.1 Relationship of Several Theories

2.2.2 Fuzzy Versus Crisp Sets

With classical or crisp sets, an element is either in the set or not in the set. A fuzzy set allows for elements to be somewhat in the set. The theory of fuzzy sets has been examined and developed extensively over the past 35 years. Key work has been accomplished by Zadeh (1968, 1971, 1978, 1979), Gaines (1978), Klir (1988), and Yager

(1993). For each operation in classical set theory a counterpart exists in fuzzy set theory. Union, intersection, complement, and Demorgan's Laws all hold for fuzzy sets.

When researchers deal with uncertainty, probability and fuzzy sets have often been presented as distinct theoretical foundations. Yet when one examines the underlying axioms of both probability and fuzzy set theories, the two differ only in one axiom in a total of sixteen axioms needed for a complete foundation (Ross, 1995). The axiom is the axiom of excluded middles. In probability theory the probability of A and $\sim A$ (not A) is 0, the probability of A or $\sim A$ is 1. It is not necessary for these conditions to hold in fuzzy set theory. The implication of fuzzy sets not following the axiom of the excluded middle is that given evidence for the membership of an event into a fuzzy set does not necessarily give evidence that the complement of the event is not in the set.

2.2.3 Applications of Fuzzy Set Theory in Reliability

Fuzzy set theory has been used in connection with reliability estimates in many ways (Bowles and Palaez, 1995). The term *fuzzy fault tree analysis* is quite common in the recent literature on fault tree analysis (Weber, 1994; Dunyak and Wunsch, 1998; Dunyak, Saad, and Wunsch, 1999; Guimarees and Ebecken, 1999; Pan and Yun, 1997; Suresh, Babar, and Raj, 1995; Tanka, *et al.* 1983). The idea is that when there is not enough information to provide probabilities at each node in a fault tree, then one must incorporate expert knowledge in the form of fuzzy sets or "fuzzy probabilities." The idea of fuzzy Markov models applied to robot reliability is considered by Leuschen *et al.* (1998) as a technique for analyzing fault tolerant designs under "considerable uncertainty."

Quantifying linguistic information to be incorporated into a problem is one of the most exploited uses of fuzzy logic. Fuzzy sets have been used to incorporate linguistic information into the survivability analysis of protective structures (Wong, Ross, and Boissonnade, 1987). They have been used to incorporate linguistic information into fuzzy Markov models (Ramachandran, Sankaranarayanan, and Seshasayee, 1992). Fuzzy set theory has been used in the definition of failures when defining a system as fully failed or fully functioning (i.e., a binary failure state) is not acceptable (Cai, Wen, and Zhang, 1993). Linguistics can be set up to quantify the amount of failure present. As with the present study, fuzzy sets have also been used in the analysis of type I censored data (Kanagawa and Ohta, 1992; Hwang and Yeh, 1994).

2.3 Bayesian and Fuzzy Bayesian Methods

In Bayesian statistics, expert knowledge is introduced in the form of prior distributions. The *a priori* information is essentially the analyst's assumption about the shape of the data. The analyst makes their assumptions based on the phenomena at hand in conjunction with passed experience, expert opinion, and knowledge of probability theory prior to observing any data. A probability distribution is assumed; then the posterior distribution is found using Bayes rule on the observed data given the prior distribution.

Let X be a random variable with probability density $f(x, \theta)$. In the Bayesian point of view, θ , the realization of the parameter to be estimated, is considered a random variable with density $f_\Theta(\theta)$. $f_\Theta(\theta)$ is determined by the analyst to express the prior belief before any data is observed, that is, *a priori*. $f(x, \theta)$ is called the conditional density of X, given $\Theta = \theta$, and is rewritten as $f_{X|\Theta}(x, \theta)$. Then the joint density of X and Θ is given by

$f_{X,\Theta}(x, \theta) = f_{X|\Theta}(x, \theta) * f_\Theta(\theta)$. The Bayesian approach can be considered an updating process (Hoyland and Rausand, 1994). An initial assumption is made about the density for Θ before any data is observed. This prior distribution is then updated to the posteriori distribution. The observed value of X has therefore changed the belief of the value of Θ.

The Bayesian methodology has been applied in cases of small, censored samples (Ditlesven, Tarp-Johansen, and Denver, 2000). In this case, it is recommended that the small censored data set be supplemented by a priori knowledge taken from the field at the time of the data acquisition. Thus the prior distribution is based on the conditions from which the data were gathered. Ditlevsen and Vrouwenvelder (1994) use Bayesian inference for totally censored data. The development is made under the assumption of Gaussian data.

Bayesian analysis has been used to estimate sample size and censoring time based on operational time (Sharma and Krishna, 1993). Coolen (1996) takes into consideration the problem of studying censored data in reliability problems. In this paper Bayesian analysis is used to incorporate expert knowledge from the judgment of engineers when faced with a shortage of data. Bayes estimation has been used in the realm of regression. Sarhan (1999) uses Bayes estimators to find coefficients a and b in the general hazard rate model $h(t) = a+bt^{c-1}$. The Bayes estimators under the squared error loss function are derived when a and b are assumed to have prior exponential distributions. This estimate is then compared with the least squares regression estimate.

2.3.1 Fuzzy Bayesian Methods

The term "Fuzzy-Bayesian" shows up in many different ways in the literature. This term seems to be used whenever fuzzy sets are used in conjunction with Bayes rule.

If fuzzy sets are used to quantify qualitative information or a fuzzy number is used when faced with imprecise data, and the "fuzzy data" are used in conjunction with Bayes rule the term is used.

Fuzzy life-tests which treat the lower limit of desirable and upper limit of undesirable life characteristics as fuzzy numbers is considered by Kanagawa and Ohta (1992). The fuzzy membership functions are in the form of polynomials, and Bayes theorem is applied to obtain the posterior density of accepted and rejected lots. The development is made with the assumption of an inverted Gamma distribution as the prior distribution, and the result is that the analysis using the fuzzy sets is more in accord the analyst's "feelings."

Fuzzy-set theory allows one to quantify qualitative information and incorporate the result into the Bayesian analysis (Yang and Cheung, 1995). Chou and Yuan (1993) use this "Fuzzy-Bayesian" approach to develop an algorithm to compute the posterior probability of structure safety based on the visual inspection of existing structures. The algorithm allows for the inspector to report information such as minor crack, significant crack, or major crack with respect to the condition of a support beam. This information is then described by a fuzzy set, then used in the Bayesian Analysis. In another work involving structure reliability, Savchuk (1993) uses fuzzy numbers to describe vagueness in the data and then applies these values in a Bayesian analysis. Then this approach is compared with one based purely on a Bayesian approach. The conclusion in the study is that both approaches produce similar results. The use of fuzzy numbers has a simpler form as a key advantage.

2.4 The Bootstrap Method

Subsampling techniques are used to get a rough estimate of variability of a statistic and homogeneity of the sample. These methods involve taking subsamples from the original sample. Mahalanobis used subsamples for assessing variability under the name "interpenetrating samples" in the early 1940's (Hartigan, 1969). In 1958, Professor John Tukey of Princeton University developed a method for reducing bias and estimating standard errors, which he named the "jackknife." The "jackknife" method came from an idea first proposed by Quenouille for estimating the bias of an estimator and is sometimes credited to him (Efron and Tibshirani, 1993).

Given a random sample of size n from an unknown distribution, the jackknife estimate of the standard error consists of taking n "jackknife samples" from the original sample. These samples are taken leaving out the i^{th} observation from the original sample. Then, the statistic of interest is recomputed using this subsample and is called the i^{th} jackknife replication for $i = 1, \ldots, n$. The jackknife estimate of the standard error is then the square root of $\frac{n-1}{n}$ times the sum of the squared difference of each jackknife replication from the mean of all the jackknife replications.

In the late 1960's and early 1970's, Hartigan of Princeton and Yale Universities wrote several papers which brought some mathematical rigor to the idea of subsampling for use in error analysis. He proved several essential theorems for exact and approximate results (Hartigan, 1971) . In 1975 he published a paper on some asymptotic properties of subsampling, including a generalization of the central limit theorem. All this rigor opened the door Efron's Bootstrap in 1979.

The bootstrap method was introduced by Professor Efron of Stanford University in 1979 as a computer based way of estimating the standard error of an estimator (Efron, 1979). The method consists of resampling with replacement from the original sample. Given a sample of size n from an unknown distribution, the method requires taking a sample of size n with replacement from the original sample. Since the number of possible bootstrap samples are so numerous with $n > 10$ say, only a random sample of them is taken. Thus, given a random sample of size n from an unknown distribution, let $\hat{F}$ be the empirical distribution putting probability of $\frac{1}{n}$ on each of the n observations. A bootstrap sample is then a random sample of size n drawn with replacement from $\hat{F}$. Thus, some of the original sample does not appear, some appear once, and some appear more than once.

The bootstrap method may be used on any parameter estimate or statistic, and no theoretical calculations are required, no matter how mathematically complex the estimator may be. Like the jackknife, for each of the bootstrap samples the statistic of interest is recomputed. These are called the bootstrap replications. The bootstrap estimate of the standard error is then the square root of $\frac{1}{n-1}$ times the sum of the squared difference of each bootstrap replication from the mean of all the bootstrap replications.

This method can be used on the mean, trimmed mean, median, or any other statistic. The non-parametric version requires no assumptions about the distribution of the data. It can be used to find standard errors of the statistic, bias in the estimate, and confidence intervals for the estimate, each of which are discussed below. The non-parametric version "resamples" from the data. The parametric version uses the data to estimate parameters of a distribution and then takes samples from the parametric distribution. Instead of simulating the sample distribution of a statistic estimating a parameter, the Bayesian bootstrap simulates the posterior distribution of the parameter (Rubin, 1981). Lo (1987) provides an asymptotic justification for the Bayesian bootstrap.

If the data set is small and/or it contains "bad" values, it may be better to estimate the average of the data using the median. The bootstrap method allows us to estimate confidence intervals for the median and helps to minimize the effect of a bad data point (Pokorny, 1996). Since order statistics are not smooth functions, confidence intervals tend to be more conservative for the median than the mean in many situations. The following subsections contain a more mathematical explanation of the bootstrap and details how to obtain the standard error, bias, and confidence intervals using the bootstrap method.

2.4.1 Standard Error Estimates

Given a random sample of size n from an unknown distribution F (i.e. $F \rightarrow \mathbf{x} = (x_1, x_2, \ldots, x_n)$) let $\hat{F}$ be the empirical distribution putting probability of 1/n on each x_i, $i = 1, 2, \ldots, n$. A bootstrap sample is a random sample of size n drawn with replacement from $\hat{F}$, that is $\hat{F} \rightarrow \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$.

If estimating a parameter $\theta = t(F)$ based on the sample $\boldsymbol{x}$, denoted $\hat{\theta} = s(\boldsymbol{x})$, the statistic computed using a bootstrap sample is called a bootstrap replication of $\hat{\theta}$, $\hat{\theta}^* = s(\boldsymbol{x}^*)$. The bootstrap algorithm is a computational way of obtaining an approximation to the numerical value of the standard error of $s(\boldsymbol{x})$.

The Non-Parametric Bootstrap Algorithm

1. Take B bootstrap samples $\boldsymbol{x}^{*1}, \boldsymbol{x}^{*2}, \ldots, \boldsymbol{x}^{*B}$, each consisting of n data values drawn with replacement from $\boldsymbol{x}$.

2. Evaluate the bootstrap replication corresponding to each bootstrap sample, $\hat{\theta}^*(b) = s(\mathbf{x}^{*b}), \quad b = 1,2,\ldots, B$

3. Estimate the standard error $se_F(\hat{\theta})$ by the sample standard deviation of the B replications (Efron and Tibshirani, 1993):

$$se_B = \left\{ \frac{1}{B-1} \sum_{b=1}^{B} \left(\hat{\theta}^*(b) - \hat{\bar{\theta}}^* \right)^2 \right\}^{\frac{1}{2}} \text{ where } \quad \hat{\bar{\theta}}^* = \frac{1}{B} \sum_{b=1}^{B} \hat{\theta}^*(b) \tag{13}$$

The parametric bootstrap takes B samples of size n from the parametric estimate of the population $\hat{F}_{par}$ based on the original data, then proceeds with steps 2 and 3 above. The parametric bootstrap tends to produce better confidence intervals in most situations but, the assumption of the underlying distribution must be correct. In contrast, the non-parametric bootstrap is asymptotically efficient. That is, for large samples they give accurate answers regardless of the underlying population.

Efron (1981) shows that the bootstrap estimate of the standard error works not only for Type I censored data but also with the more complex situation of random (Type

IV) censoring. With censored values a data point is denoted (t_i, d_i),

where $d_i = \begin{cases} 1 & \text{if } x_i \text{ is uncensored} \\ 0 & \text{if } x_i \text{ is censored} \end{cases}$.

Now, bootstrap samples can be taken by randomly sampling the points (t_i, d_i). It is also possible to sample the data separately (i.e., randomly selecting a t_i and then a d_i). The later approach has the effect of considering the censoring to be random. It has been shown that with large samples of right-censored and left-truncated data both approaches are equally effective (Bilker, 1997).

2.4.2 Bias Estimates

The bias of a bootstrap estimate can be estimated in several ways. Two methods are particularly interesting, in which the first is much easier to obtain, but the second provides a better estimate for some cases. Given $\hat{\theta}$ is estimating θ the bias is defined as the difference between the expectation and the value of the parameter θ (Lehmann, 1983). Thus we have $bias_F$ = $E_F[s(\boldsymbol{x})]$-t(F), and the "simple method" to calculate the bias is

$bias_B = \hat{\hat{\theta}}* - \hat{\theta}$.

The second method considers the proportion that each data value was used in the bootstrap samples. Let $\mathbf{P}^*$ be the proportion of a bootstrap sample that equals the j^{th} original data point. The resampling vector $\mathbf{P}^* = (P_1^*, P_2^*, \ldots P_n^*)$ has non-negative components summing to one. Now each bootstrap sample, $\boldsymbol{x}^{*\text{b}}$ has a corresponding resampling vector $\mathbf{P}^{*\text{b}}$. Let $\bar{\mathbf{P}}* = \frac{1}{B}\sum_{b=1}^{B}\mathbf{P}^{*b}$. The bias in the estimate is then

$\overline{\text{bias}}_B = \bar{\hat{\theta}}^* - s(\bar{\mathbf{P}}^*)$. A rule of thumb is that a bias of less than 0.25 standard errors can be ignored (Efron and Tibshirani, 1993).

2.4.3 Confidence Intervals

Chen and Lo (1996) show that the bootstrap approximation for Kaplan-Meier estimator using the Student's-t distribution provides confidence intervals with better coverage probability than the normal approximation. Prediction curves with confidence intervals based on censored data can be created using a modified bootstrap method (Escobar and Meeker, 1999).

It is recommended to use the BC_a (bias-corrected and accelerated) method for bootstrap confidence intervals (Efron and Tibshirani, 1993). This method is based on the same concepts as standard statistical methods that use percentiles from the Gaussian or Student's t-distributions. With this method, percentiles of the bootstrap estimates are used.

Let $\hat{\theta}^{*(\alpha)}$ be the 100αth percentile of the bootstrap replications. The BC_a interval of intended coverage 1-2α is given by $\left(\hat{\theta}^{*(\alpha_1)}, \hat{\theta}^{*(\alpha_2)}\right)$, where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(\alpha)}}{1 - \hat{\alpha}\left(\hat{z}_0 + z^{(\alpha)}\right)}\right) \text{ and } \alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\alpha)}}{1 - \hat{\alpha}\left(\hat{z}_0 + z^{(1-\alpha)}\right)}\right) \tag{14}$$

The components that make up α_1 and α_2 are detailed in the following paragraphs. $\Phi(\cdot)$ is the standard normal cumulative distribution and $z^{(\alpha)}$ is the 100αth percentile point of the standard normal distribution. The following is employed

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\#\left\{\hat{\theta}^*(b) < \hat{\theta}\right\}}{B}\right) \tag{15}$$

to calculate $\hat{z}_0$ as the proportion of bootstrap replications less than the original estimate $\hat{\theta}$. This is roughly a measure of the median bias of the estimate. Thus, $\hat{z}_0$ is the bias correction. The acceleration $\hat{\alpha}$ is calculated by

$$\hat{\alpha} = \frac{\sum_{i=1}^{n}\left(\hat{\theta}_{(.)} - \hat{\theta}_{(i)}\right)^3}{6\left\{\sum_{i=1}^{n}\left(\hat{\theta}_{(.)} - \hat{\theta}_{(i)}\right)^2\right\}^{3/2}} \tag{16}$$

where $\hat{\theta}_{(i)}$ is the estimate based on the original sample with the i^{th} point (x_i) deleted, and $\dot{\theta}_{(\,)} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{(i)}$. The idea of deleting the i^{th} element is the jackknife estimate, which is the concept on which the development of the bootstrap was based. Notice that if $\hat{\alpha}$ and $\hat{z}_0$ are both zero then $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. Under this condition it is the original percentile method. The BC_a has been shown to produce good results in some situations with censored data (Efron, 1981).

The acceleration refers to the rate of change of the standard error of $\hat{\theta}$ with respect to the true parameter θ. The standard normal approximation assumes that the standard error is the same for all θ. The acceleration constant $\hat{\alpha}$ corrects for deviations from this assumption. Essentially, the BC_a method can be viewed as an improvement to the percentile method.

CHAPTER 3

FUZZY PRODUCT-LIMIT ESTIMATORS

3.1 Introduction

Recall that reliability estimates on a piece of equipment under one mode of operation may be available, and it is desired to use this information along with the operation time under a new mode of operation to obtain point estimates and confidence bounds for the new mode of operation. In this chapter, methods are developed that work with very small data sets ($n < 20$) that are highly censored (30% or more). This data set can be considered as the operation time under the new mode of operation.

The product-limit estimator views data from a binary perspective in that before the failure or censored time the belief of survival is one and after it is zero. In this chapter, fuzzy membership functions are developed to describe the belief in the survival of a component or piece of equipment associated with a censored time. This member ship function permits belief in survival to take on values between one and zero after the observed censoring time. A survival curve is generated from this data using the concept of the product-limit estimator, except now at each point the amount of belief in the survival of the individual units are aggregated to form the product at the given point. In addition, the algorithm allows the user to input their opinion of the mean survival time (i.e., the reliability estimate from the old mode of operation) and the amount of confidence they have in the input. This input may be the reliability estimate from the manufacturer of the new technology being implemented. The confidence level would coincide with the confidence in the manufactures testing procedures, or it may be from other sources. The input could be expert knowledge, given by engineers familiar with this type of technology, and the confidence directly corresponds to the engineers' confidence in their own opinion. It could be reliability estimates from "like" equipment, and the confidence correlates with the amount of "likeness" in the technologies. Another possibility that is often the case in many industries today is that data in the past had not been collected properly. Thus the old data are suspect, containing erroneous values, but giving a rough estimate. This rough estimate can be input along with an intuitive confidence in the estimate. For example, the average (mean, mode, or median) from this data could be input as the user's opinion with a confidence based on their confidence in the old data. This situation allows a new beginning, taking as much knowledge as possible from the old data until enough "clean" data has been collected to provide statistically significant results on its own.

Highly censored reliability data is being studied. The data sets are very small and right-censored (survival time is known, but the unit has not failed). The data can be censored 30% or more. The assumptions made about the data in the past are suspect. Since the data set is so small and the equipment new, it is uncertain as to whether the underlying distribution is exponential, Weibull, or some other distribution. With the method proposed here, the data set itself is used to extract information in the form of fuzzy membership functions about each of the data points independently, based on the available information.

Consider a case where there are five data points with censored times of 1, 3, 24, and failure times of 38 and 42 hours. Intuitively, one might think that the censored values (if allowed to continue) would fail at around 40 hours. There is more evidence that the unit that lasted 24 will last until 25 than the unit that lasted 3 will last until 25 hours. How much evidence is there that one of these units can last until 25, 38, 40, 42, or 108 hours? This question depends on the quality of the data, the unit's rank in the data set, and other factors. Several methods are developed that address this question.

How does one value influence another? The idea is that items that are alike should act alike. Two engines designed and built the same way should behave the same and fail in the same way at around the same time. This hypothesis is the basis on which warranties are made. Units are built and tested. Statistics are used to estimate MTTF, MTBF, and possibly a survival curve. When more units are built the same way and distributed to consumers, the warranties are based on the statistics of the tested units. The probability of survival is assumed to be the same for all of the units. Again, the problem is to take an old estimate (possibly from another mode of operation or like equipment) and update it with the new data. Thus, it is expected that there is a relationship between the old and new data. The strength of that relationship is considered here in the form of the user input to the model.

What is known about a censored unit is that it has survived until time t_i. Thus, evidence supports that the unit will survive until time t_i with certainty. After time t_i the evidence that the unit can continue to survive is not at all deterministic. One may consider that the units fail immediately after the censored time. This distinctly pessimistic view represents a lower bound on the available evidence of the survival of the unit. It is what is often used when dealing with censored data. Another may consider that censored units all last forever or at least as long as the scope of inference. This view is the most optimistic possibility and can be used as an upper bound on the available evidence of survival.

3.2 Trapezoidal Membership Functions

Considering the units to fail immediately after censoring or lasting through out the scope of inference as above provides lower and upper bounds for any belief function. Trapezoidal functions are one of the most commonly used membership functions in fuzzy inferences (Ross, 1995). They are easy to describe mathematically, they produce "quick and dirty" solutions when needed, and they accurately describe belief in many situations.

Now a trapezoidal membership function to describe belief about the survival of unit i censored at time t_i is considered. From time zero to t_i the membership is one. From time t_i to $t_n + \frac{i}{TOT}$ the membership function is linear, such that it is 1 at t_i and 0 at $t_n + \frac{i}{TOT}$, where t_n is the time of the largest value in the data set (censored or not) and TOT is the total operating time (sum of all times in the data set). After t_n the membership is zero. Setting membership to zero just after t_n indicates that there is little evidence of survival after this time.

Let $f_i(x)$ be the membership function for the i^{th} unit. The above description is then expressed mathematically by the following piecewise equation defined for $0 \le x \le \infty$, $0 \le i \le n$.

$$f_i(x) = \begin{cases} 1 & \text{if} \quad x \le t_i \\ \dfrac{t_n - x + \dfrac{i}{TOT}}{t_n - t_i + \dfrac{i}{TOT}} & \text{if} \quad t_i \le x < t_n + \dfrac{i}{TOT} \\ 0 & \text{if} \quad x \ge t_n + \dfrac{i}{TOT} \end{cases} \tag{17}$$

Then the membership functions for the entire data set with censored and failure times can be expressed by

$$\mu\ (x,t,i) = \begin{cases} f_i(x) \text{ if} & t_i \text{ is censored} \\ \text{otherwise} & (\text{ not censored }) \\ & \begin{cases} 1 \text{ if } x < t_i \\ 0 \text{ if } x \ge t_i \end{cases} \end{cases} \tag{18}$$

These equations are very simple, easy to describe, and they allow the censored values to express some information based on the data set. At the same time the biggest problem is that this method may be an oversimplification of the belief functions about the censored values. This version of the membership functions for the censored values takes into consideration only the largest time in the data set without regard to whether it is a failed time or censored. In the following example recall the notation used in chapter 2. A data value is given by the pair (t_i, d_i) which indicates a time (t_i) and an indicator (d_i) of whether the i^{th} time is censored (d_i=1) or not censored (d_i=0).

Consider an example of five data points generated from an *exponential*(100) distribution. The membership functions generated from $\mu(x, t, i)$ are graphed in Figure 3.1 to provide graphical description of these membership functions.

Example 1: Given 5 units with 4 censored and 1 failed then the data set {(10, 1), (50, 1), (110, 0), (240, 1), (350, 1)} has $n=5$ with 80% censored values. The graphs of the membership functions calculated using the trapezoidal function given above by $\mu(x, t, i)$ appear in Figure 3.1.

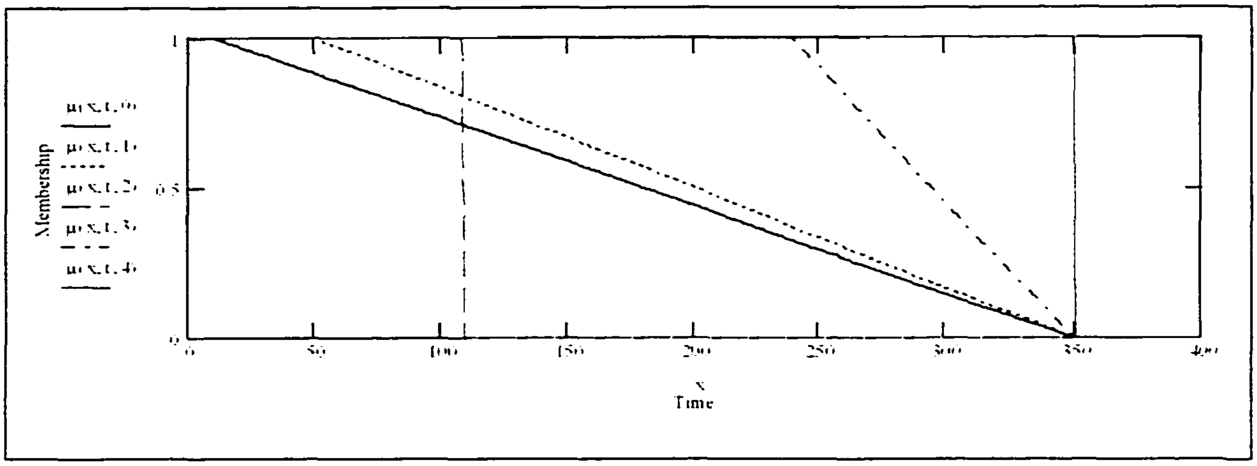


Figure 3.1 Trapezoidal Membership Functions

Considering Figure 3.1 there is approximately a 0.70 belief that the first unit will survive until time 110 and a 0.80 belief the second unit will survive to time 110. This approach is simple, easy to visualize, and it gives relatively good results for the purposes presented here. It may not completely describe the survival based on the evidence given by the data. For example, if the third unit failed at 230 instead of 110 it would have very little effect on the membership functions of the other units. This example is counterintuitive since more evidence would suggest a larger survival time under these circumstances.

In addition, this method gives essentially no belief in the survival of a unit after t_n. This lack of belief in continued survival is counter to the philosophy of this thesis. Intuitively, one may think that the censored units with times t_0 and t_1 should have a belief greater than 0.50 at time 240. These units should also have some possibility of survival after time 350 since the data shows that 40% units have survived to time 240 without failing and only 1 unit (20%) failed at time 110. Also there is some intuition that the unit with the last time should survive beyond 350 (using the above trapezoid membership gives a belief of survival of zero at 350+1/760 = 350.005). If the last data element had been a failure time then there may be very little belief of survival after this time.

3.3 Inverted Sigmoid Membership Function

From the previous discussion it is evident that membership functions are needed that describe the belief in the survival of a unit that is more in accord with the data. The development of such membership functions requires considering more of the data and other attributes of the data to build the membership function. One possibility is to create a linear piecewise membership function for each data point, by calculating the level of belief at each recorded time larger than t_i based on times, number of censor/failure times, and possibly other factors. This method creates a linear piecewise function with $n - i + 1$ pieces for the i^{th} point $i=1,\ldots, n$ and $d_i=1$. Another possibility is to create a smooth curve that describes the belief more fully and takes all the data into account. But what curve describes the evidence? In the words of Professor Weibull when proposing the Weibull distribution, "It is believed that...the only practical way of progressing is to choose a simple function, test it empirically, and stick to it as long as none better has been found" (Weilbull, 1951). With this philosophy in mind, a sigmoid function is suggested.

3.3.1 Sigmoid Functions

The goal now is to find a function that uses more information from the data and describes the belief more appropriately. Another popular form of a membership function is a sigmoid function. This is an *S*-shaped function that can be manipulated to describe certain phenomena. In the current situation, the smoothness of the sigmoid function is desirable but the shape is upside down. Consider an inverted sigmoid function that is a function in the shape of an upside-down *S*. A mathematical function of an inverted sigmoid function is of the form:

$$\mu_i(x) = \begin{cases} 1 & \text{if } \ 0 \le x < t_i \\ \dfrac{1}{1+\left(\dfrac{x-t_i}{TOT}\right)^c \exp\left(\dfrac{x-t_i}{a}\right)} & \text{if } \ t_i \le x < \infty \end{cases} \tag{19}$$

A graph of a function of this form is shown in Figure 3.2. It gives an inverted *S*-shape. It is smooth, continuous, and can be more flexible than the linear piecewise functions considered previously. This curve can be manipulated so that it holds close to one for as long as desired and can be forced to zero when needed. Values for a and c can be adjusted based on the data set such that $\mu_i(x)$ describes the belief in the survival of unit i. The solution to a and c is part heuristic, since it is expressing intuitive information, and part approximation theory from numerical analysis, since the belief is data driven.

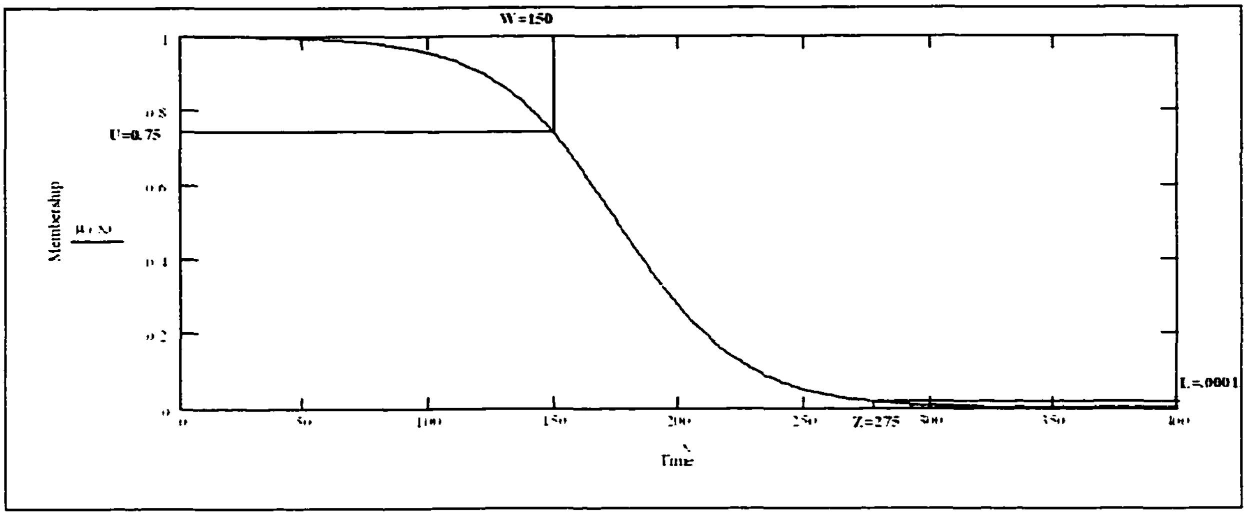


Figure 3.2 Inverted Sigmoid Function

Examining the components of this function provides more insight into why this form is chosen. The factor $\left(\frac{x-t_i}{TOT}\right)^c$ forces $\mu_t(x)$ to one when x approaches t_i. Thus $\mu_t(x)$ is continuous at t_i from the left and right. TOT scales the function to the data. Larger values of c allow $\mu_t(x)$ to stay closer to one for larger x. Smaller values of c allow $\mu_t(x)$ to decrease faster. Thus, varying c changes the slope of the function corresponding to the rate of change in the belief of survival of the unit. c as a function of the data is developed later in the current subsection.

The function $\mu_t(x)$ approaches zero as $\frac{x-t_i}{a} \rightarrow \infty$. Thus $\mu_t(x)$ can be forced to approach zero at a given point as desired by allowing a to approach zero. Now, the choice of a determines when there is no longer a belief in survival. Setting $\mu_t(z) = L$, where L is a value close to zero has the effect of bringing the right hand side of the graph to some limiting value at a given point in the time axis. Thus, L is a point where there is very little belief in the survival of the censored unit. Note that this function cannot be equivalent to zero. In addition, some upper bound, say *U*, can be specified at a given point, say *w* (i.e., $\mu_i(w) = U$). For example, there may be a desire to force the membership of a given unit to be 75% at a guess of the MTTF taken from the expert knowledge of an engineer and .01% at *TOT*. Thus, one's belief in the survival of a unit can be modeled by manipulating two points on the curve, namely (*w*, *U*) and (*z*, *L*).

Consider the following example using the exponential data from example 1. It is demonstrated in this example how the manipulation of two points can effect the graph.

Example 2: Given the data {(10,1), (50,1), (110,0), (240,1), (350,1)} the evidence may be that $\mu_2(150) = 0.75$ & $\mu_2(275) = 0.0001$. This indicates a belief of 75% that the unit that has lasted 50 will last to 150 and a 0.1% belief the unit will last to 275. This is shown in Figure 3.2.

Using $\mu_i(w) = U$ and $\mu_i(z) = L$, *a* and *c* can be defined in terms of *w*, *U*, *z*, and *L*. For $t_i \le w \le z$ and $L \le U$, this development results in the following:

$$\mu_i(w) = \frac{1}{1 + \left(\frac{w - t_i}{TOT}\right)^c \exp\left(\frac{w - t_i}{a}\right)} \tag{20}$$

and

$$\mu_i(z) = \frac{1}{1 + \left(\frac{z - t_i}{TOT}\right)^c \exp\left(\frac{z - t_i}{a}\right)} \tag{21}$$

Setting $\mu_i(z) = L$ and solving for *a* yields

$$\frac{1}{1+\left(\frac{z-t_i}{TOT}\right)^c \exp\left(\frac{z-t_i}{a}\right)} = L \tag{22 a}$$

$$\Leftrightarrow \qquad \left(\frac{z-t_i}{TOT}\right)^c \exp\left(\frac{z-t_i}{a}\right) = \frac{1}{L} - 1 \tag{22 b}$$

$$\Leftrightarrow \qquad \exp\left(\frac{z-t_i}{a}\right) = \frac{1-L}{L\left(\frac{z-t_i}{TOT}\right)^c} \tag{22 c}$$

$$\Leftrightarrow \qquad \frac{z-t_i}{a} = \ln\left(\frac{1-L}{L\left(\frac{z-t_i}{TOT}\right)^c}\right) \tag{22 d}$$

$$\Leftrightarrow \qquad a = \frac{z-t_i}{\ln\left(\frac{1-L}{L\left(\frac{z-t_i}{TOT}\right)^c}\right)} \tag{22 e}$$

$$\text{or} \qquad a = \frac{z-t_i}{\ln\left[\left(\frac{1-L}{L}\right)\left(\frac{z-t_i}{TOT}\right)^{-c}\right]} \tag{22}$$

Considering $\mu_i(w) = U$,

$$\Leftrightarrow \qquad \frac{1}{1+\left(\frac{w-t_i}{TOT}\right)^c \exp\left(\frac{w-t_i}{a}\right)} = U \tag{23 a}$$

and substituting for a from above yields

$$\frac{1}{1+\left(\frac{w-t_i}{TOT}\right)^c \exp\left(\frac{(w-t_i)\ln\left[\left(\frac{1-L}{L}\right)\left(\frac{z-t_i}{TOT}\right)^{-c}\right]}{z-t_i}\right)} = U \tag{23 b}$$

Now solving for c,

$$\left(\frac{w-t_i}{TOT}\right)^c \exp\left(\frac{(w-t_i)\left(\ln\left(\frac{1-L}{L}\right)+c\ln\left(\frac{TOT}{z-t_i}\right)\right)}{z-t_i}\right) = \frac{1-U}{U} \tag{23 c}$$

$$\Leftrightarrow \quad c\ln\left(\frac{w-t_i}{TOT}\right) + \left(\frac{(w-t_i)\left(\ln\left(\frac{1-L}{L}\right)+c\ln\left(\frac{TOT}{z-t_i}\right)\right)}{z-t_i}\right) = \ln\left(\frac{1-U}{U}\right) \tag{23 d}$$

$$\Leftrightarrow \quad c(z-t_i)\ln\left(\frac{w-t_i}{TOT}\right) + (w-t_i)\left(\ln\left(\frac{1-L}{L}\right)+c\ln\left(\frac{TOT}{z-t_i}\right)\right) = (z-t_i)\ln\left(\frac{1-U}{U}\right) \tag{23 e}$$

$$\Leftrightarrow c\left((z-t_i)\ln\left(\frac{w-t_i}{TOT}\right)+(w-t_i)\ln\left(\frac{TOT}{z-t_i}\right)\right) = (z-t_i)\ln\left(\frac{1-U}{U}\right)-(w-t_i)\ln\left(\frac{1-L}{L}\right) \tag{23 f}$$

$$\Leftrightarrow \quad c = \frac{(z-t_i)\ln\left(\frac{1-U}{U}\right)-(w-t_i)\ln\left(\frac{1-L}{L}\right)}{(z-t_i)\ln\left(\frac{w-t_i}{TOT}\right)+(w-t_i)\ln\left(\frac{TOT}{z-t_i}\right)} \tag{23}$$

Incorporating the result of a into $\dfrac{1}{1+\left(\dfrac{x-t_i}{TOT}\right)^c \exp\left(\dfrac{x-t_i}{a}\right)}$ gives

$$\frac{1}{1+\left(\dfrac{x-t_i}{TOT}\right)^c \exp\left(\dfrac{x-t_i}{z-t_i} * \ln\left[\left(\dfrac{1-L}{L}\right)\left(\dfrac{TOT}{z-t_i}\right)^c\right]\right)} \tag{24 a}$$

$$= \frac{1}{1+\left(\dfrac{x-t_i}{TOT}\right)^c \left[\left(\dfrac{1-L}{L}\right)\left(\dfrac{TOT}{z-t_i}\right)^c\right]^{\frac{x-t_i}{z-t_i}}} \tag{24}$$

The final form of the inverted sigmoid function that will be used for the membership functions herein is then:

$$\mu_i(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < t_i \\ \dfrac{1}{1+\left(\dfrac{x-t_i}{TOT}\right)^c \left[\left(\dfrac{1-L}{L}\right)\left(\dfrac{TOT}{z-t_i}\right)^c\right]^{\frac{x-t_i}{z-t_i}}} & \text{if} \quad t_i \le x < \infty \end{cases} \tag{25}$$

where c is defined above. Now that a and c are in terms of w, U, z, and L these values need to be determined to reflect the evidence for the survival of a censored unit.

3.3.2 Determining w, U, z, and L

Determining these values will complete the development of the inverted sigmoid membership functions. This determination is heuristic and based on intuition, yet the intuition is based on evidence given by characteristics of the data. Therefore, the values of w, U, z, and L are developed from an intuitive point of view in this section, and asymptotic properties of the resulting functions are considered in section 3.5.

The belief is a data-generated belief. In other words, the belief in the survival of a given unit is based on looking at the data set and the given unit's placement in the data set relative to other units in the data set. How should w, U, z, and L be selected? The values of U and L may be chosen somewhat arbitrarily because they are dependent on the values of w and z respectively. The only criteria is that U is greater than 0.5 and L is less than 0.5. The following development is made with $L = 0.0001$. This is simply a value "close" to zero. If it is not sufficiently close to zero, then two possibilities exist for the correction. The first is to define L to be closer to zero. The other possibility is through the adjustment of z such that a belief of this magnitude is plausible at $x = z$. Again, since the value of U depends on w, it too is chosen as a fixed value, and w is adjusted accordingly. In this dissertation the development is made with $U = 0.65$. Thus the development is based on finding the point w on the time axis at which the membership function is 65%. The value of U is arbitrary since the development of w could have been made around $U = 0.99$. And, the development made can be used with other values of U. Thus, the values of L and U are parameters that may be adjusted or "tweaked" to fit a given set of circumstances. The effect of changing U is considered after the development of w and z. After the development, different values of U change the amount of optimism in continued survival of a censored unit. Values close to 1.0 indicate more optimism for continued survival and for values close to 0.5 indicate less optimism for continued survival.

Let w_i be the value of w for the i^{th} unit. First, it is noted that $w_i > t_i$ because for all values less than or equal to t_i the belief of survival is one. When considering the belief in the survival of the i^{th} unit, one factor is the total time of the units that survived to time t_i and beyond. There are several motivations for this approach. The first concerns the MLE of λ for Type I censoring when an exponential distribution is assumed. The total time on test is required, which in the current situation takes into consideration an estimate of λ based on the data that survived to time t_i and beyond. Thus, a possibility for w_i is $w_i = t_i + \sum_{j=i}^{n} t_j$. But, this value grows too quickly and in general is an over optimistic belief in the survival of a given unit.

Considering the data from Example 2, $w_1=10+760=770$, $w_2=50+750=800$, $w_4=240+590=830$, $w_5=350+350=700$. This result says that there is a 65% belief that the unit that lasted until time 10 will last until time 770, more than twice that of the longest recorded time in the data set, a rather optimistic belief. Also, this is 10% longer than the unit that has lasted until time 350. The result can be brought back into proportion by replacing the sum by the average of the units that survived to time t_i and beyond but, the last unit would still have the overly optimistic belief of $w_n = 2t_n$. Thus a better choice is to divide each sum by n. This division by n allows the early censored times to have high belief for a longer time and the censored values toward the end of the data set to be less optimistic of continued survival. Now, $w_i = t_i + \frac{1}{n}\sum_{j=i}^{n} t_j$ gives plausible results for a 65% belief of the survival of a censored value. When considering the data from Example 2, w_1=10+760/5=162, w_2=50+750/5=200, w_4=240+590/5=358, w_5=350+350/5=420. In this example t_1 has a 65% belief of survival more than fifteen times its censored time, t_2 three times its censoring time, t_4 about ½ its censoring time, and t_5 about 1/5 its censoring time. This result fits intuition in that for a unit censored early, there is stronger evidence of survival further out on the time axis, and for units censored later not much evidence suggests that they may continue to survive. At the same time there is no evidence that they may fail. This evidence in the data set is the motivation for using fuzzy sets (i.e. *belief(survival)* + *belief(~survival)* < 1). It is the concept of ignorance that says there is some belief in survival without implying the amount of belief in failure.

This development still lacks information that can be obtained from the data set. That is the information gained by knowing the number (or proportion) of censored data relative to failed values. If in the above example there were ten data points before time 10, the censoring status of those units would not affect the membership functions of the data points (10,1), (50,1), (240,1), and (350,1). Although, if all of the added times were fail times, the belief of survival for the censored units would not extend much further beyond their current censored times because this is evidence of an early fail time. But, if the added times were censored times, then the current values of w_i are still plausible. Therefore, let *cen* be the number of censored values in the data set then, in the current addition to t_i the factor $\dfrac{cen+n}{n+n}$ scales the amount of optimism by the number of data that is censored, resulting in:

$$w_i = t_i + \left(\frac{cen+n}{2n}\right)\frac{1}{n}\sum_{j=i}^{n} t_j \tag{26}$$

Continuing the example, $w_1 = 10 + \frac{4+5}{2 * 5^2} * 760 = 146.8$, w_2=185, w_4=346.2, w_5=413. The result is that 90% of what was added without consideration of the amount of censored data is added to the censored times. This result is reasonable since there is not as much optimism about the survival of a censored unit when a failure has been observed.

The development of *w* is only half of the picture. The value of *z* plays an important roll for the shape of the membership function even at times less than *w*. Consider Figure 3.3, with t_1=t_2=10, w_1=w_2=200, z_1=300, and z_2=800. The graphs both have $\mu(x)$=1 at *x*=10 and $\mu(x)$=0.65 at *x*=200, but notice the effect that changing *z* from 300 to 800 has. The membership function dropped in value for *x* less than *w* =200 when *z* is increased. Thus, lowering the value of *z* may raise the belief of survival for values less than *w* and decrease the belief in survival for values greater than *w*.

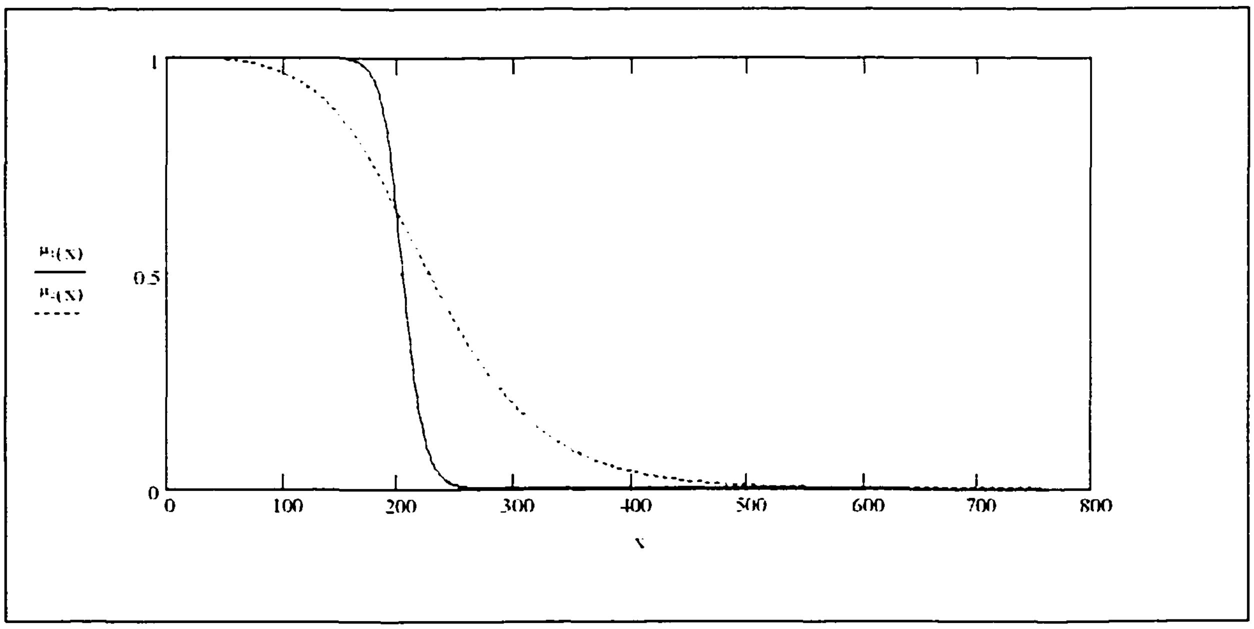


Figure 3.3 Effects of Changing *z*

Now, z is a value of x for which there is little belief in survival of the unit. It is noted that z_i must be greater than w_i. Recall that when assuming an exponential distribution, the MLE of λ for Type I censoring is $\lambda = \frac{f}{\sum_{i=1}^{n} t_n}$, where f is the number of failures in the data set, and the MTTF can be estimated using λ^{-1}. This estimate is biased for small samples but behaves asymptotically (Hoyland and Rausand, 1994; Mann, Schafer, and Singpurwalla, 1974). However, large sample theory indicates that this method is appropriate when sufficient data is available. The bias causes overestimates, occuring when n is small and/or f is small. Since small and highly censored data sets are under consideration in this study, a good heuristic is offered as $z_i=w_i+TOT/(f+1)$. This value of z_i seems to be overly optimistic since for small and highly censored samples TOT/f is overly optimistic itself, and it is suggested to add to it w_i. Is it overly optimistic for a membership function that expresses intuition? Consider Figure 3.3. If there are no failures after the i^{th} censored time, then $\mu_2(x)$ expresses the belief. In $\mu_2(x)$, z is 800, but considering the graph, it is seen that there is very little belief of survival after time 450. Also it is not as optimistic as $\mu_1(x)$ for values less than 200. Now, if there is a failure after the i^{th} unit, then it provides information in regard to the survival time. With this fact, the belief of survival is high that the i^{th} unit can survive until the time of the observed failure but depending on the other data, the belief of survival drops after this time. This belief is accurately modeled by $\mu_1(x)$. With these concepts stated, let f_i indicate the number of failures that occur after time t_i. Then,

$$z_i = w_i + \frac{TOT}{f_i + 1} \tag{27}$$

It is noted that z_i is directly related to w_i. Thus the belief described by the membership functions is based on the following evidence from the data set. The censored time t_i plays a critical role in both w_i and z_i. The number of data points, the total operating time, the proportion of censored times, the total amount of operating time observed from units surviving to t_i and beyond, and the number of failures that occurred after time t_i all provide information to form the shape of the membership function for a unit censored at time t_i.

This choice for z and L pins down some limiting point where the evidence for continued survival is very small. Other possibilities exist. In the above development, all data values between the j^{th} and $j+1^{st}$ failure tend to the same limiting point (z_i, L). At first glance, the two points having the same limiting point seems to be at odds with the essential idea in this development, that each data point is influenced by the other data and the fuzziness is unique for each data point. For this method the differences are achieved by way of the values of w and U.

Now that w is determined based on the data, different values of U can be considered. Changing the parameter U between 0.5 and 0.99 has the effect of providing more optimism in the continued survival of a censored unit at larger values and less optimism for smaller values. Considering the exponential data from Example 1 {(10, 1), (50, 1), (110, 0), (240, 1), (350, 1)}, Figure 3.4 shows the membership functions for the data when $U = 0.50$, Figure 3.5 shows the membership functions for the data when $U = 0.65$, and Figure 3.6 shows the membership functions for the data when $U = 0.99$.

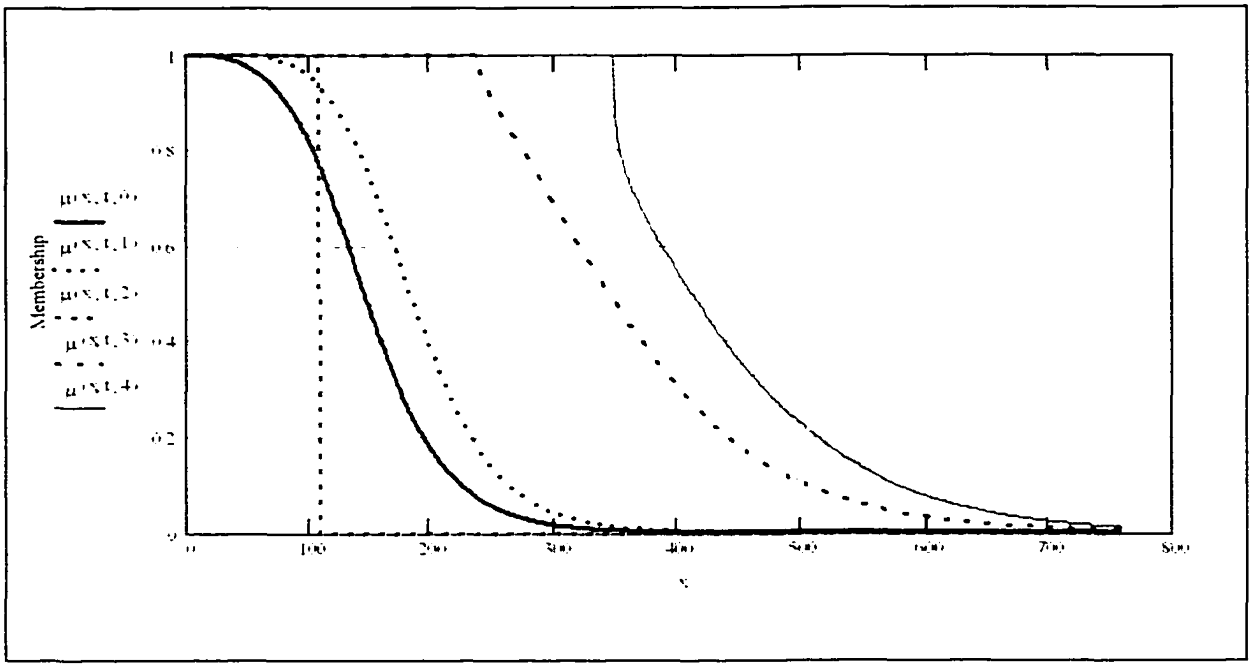


Figure 3.4 $U = 0.50$ Membership Functions

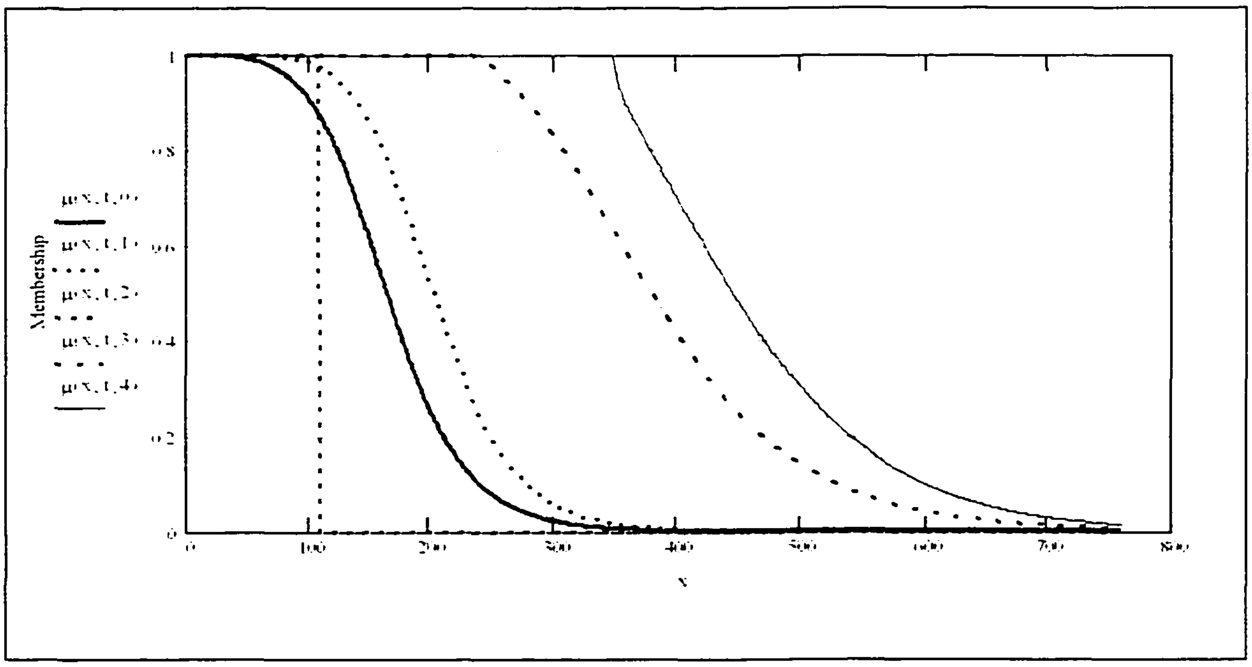


Figure 3.5 $U = 0.65$ Membership Functions

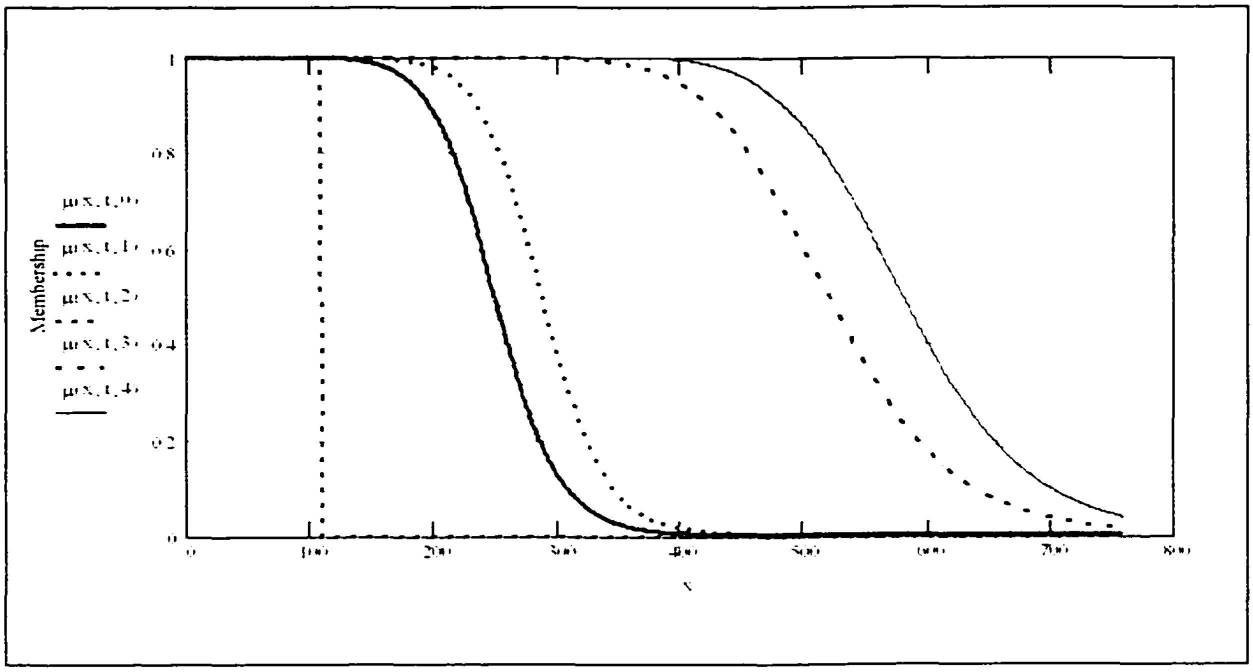


Figure 3.6 $U = 0.99$ Membership Functions

This development of membership functions is intuitive, based on evidence derived from the data. After the development of the Fuzzy Product-Limit estimator of the survival curve, properties of the membership functions are considered in section 3.5. The next section is concerned with incorporating into the model the input of a user's expert opinion and confidence level as described in section 3.1.

3.3.3 Incorporating Expert Knowledge

As stated in section 3.1 a useful method should allow for input, in the form of an external "opinion" of the mean survival time. A confidence level of this opinion is also taken as input. This opinion can be from expert knowledge, reliability estimates from the manufacturer of the equipment or "like-equipment", or reliability estimate taken from old company data. Let *ek* be the user's opinion of the mean survival time, and let *uc* be the user's confidence level in the opinion. This information is then used to adjust the membership functions accordingly. Again, adjustments made to w_i and z_i are all that is needed to manipulate the membership function. Any changes in *U* are parameter changes in the estimator and are considered separately.

There exist two cases when considering the censored data in relation to the user's opinion. The first case is that the censored value $t_i > ek$. In this case the membership function should be reduced, in some proportion to the confidence level (*uc*), to be closer to t_i. If the user had a 95% confidence level, say, then the membership function for t_i should be very close to a step function going from one to zero at t_i. Because it is "very strongly" believed that the unit has surpassed the expected survival time and the belief for continued survival is bleak the function should approach a step function around t_i. In terms of w_i and z_i, z_i should approach w_i and w_i should approach t_i as the user's confidence approaches 100%.

The second case is if the censored value $t_i \leq ek$. In this case, as user's confidence approaches 100%, the membership function for t_i should approach a step function with the step from one to zero at *ek* since the belief is that the survival time is at *ek*. This step function is accomplished by letting z_i approach w_i and w_i approach *ek* as *uc* approaches 100. To reiterate censored times that are larger than *ek* have membership functions reduced to around the censored time and censored times less than *ek* have membership functions extended to form around *ek*.

ek can be any non-negative real number, since it is a survival time it cannot be negative. *uc* will be taken as input in the range of 0 to 100. It is noted that if the user's confidence is 100% then this system is not required because the user already knows with certainty the survival time. For all practical purposes, there is no need to use the method if the user's confidence is greater than 95%. In what is developed in this dissertation, if the user's confidence (*uc*) is 100%, a division by zero occurs. In theory this estimator is developed for *uc* to approach infinitesimally close to 100%. Also, a confidence of 0% indicates the user has no confidence in *ek*; therefore, this value lends nothing to the estimate. Again, for all practical purposes a user's confidence should be at least 5%.

The relation

$$conf = \frac{n * uc}{100 - uc} \tag{28}$$

allows the user confidence to be in terms of values from 0% to 100% and allows the following discussion to flow in a more intuitive manner. In addition, this confidence value is scaled to the number of data in the data set (*n*).

Consider the case $t_i > ek$. It is desired that, z_i should approach w_i as the user's confidence approaches 100%. Starting with equation (27), simply multiplying the term $TOT/(f_i + 1)$ by $n/(n + conf)$ has the desired effect. As *conf* approaches zero, the factor $n/(n + conf)$ approaches one, and as *conf* approaches infinity, the factor $n/(n + conf)$ approaches zero. Thus, as *conf* approaches zero the membership function is less effected and as confidence grows z_i approaches w_i. Performing simple algebra reveals that $n/(n + conf) = (100 - uc)/100$, which is the inverse of the user's confidence (i.e., the term $TOT/(f_i + 1)$ is reduced inversely proportional to the user's confidence). Therefore,

$$z_i = w_i + \frac{TOT * n}{(f_i + 1) * (n + conf)} \tag{29}$$

provides the desired result.

It is now desired to have w_i approach t_i as the confidence level increases. This convergence is easily achieved by dividing by $2n(n+conf)$ instead of $2n^2$ in the second term in equation (26). The effect is the same as that for z_i, the addition to t_i is reduced inversely proportional to the user's confidence. The resulting equation is:

$$w_i = t_i + \frac{cen + n}{2n\ (n + conf)} \sum_{j=i}^{n} t_j \qquad (30)$$

Attention is now turned to the case of $t_i \leq ek$. In this situation, it is desired that z_i approach w_i and w_i approach ek as uc approaches 100. With this motivation, z_i as defined in equation (29) provides the desired result. Next, w_i needs to be defined so that it approaches ek as uc approaches 100. The goal in this case is that the censored time, t_i, have less weight and the user's input, ek, have more weight as the confidence level rises.

Starting with equation (30) consider the factor $\frac{cen + n}{n + n}$. This is a ratio of the number of censored values to the total number of values that has been tempered by the addition of n in the numerator and denominator. The result of this addition of n to the numerator and denominator is that a small proportion of censored values to failures is not as severely penalized. If $conf$ is added to the numerator and denominator the term $\frac{cen + n + conf}{2n + conf} \sum_{j=i}^{n} t_j$ is also tempered by the user's confidence and the data set has less of an impact on this factor as $conf$ increases ($conf$ is scaled to the number of data, n).

Next, consider $\frac{n}{n + conf} t_i$. If $conf$ is zero, then the term is t_i and as $conf$ increases the term tends to zero. Some algebra yields, $\frac{n}{n + conf} = \frac{100 - uc}{100}$, the inverse of the user's confidence as a percentage. To allow *ek* more weight as the user's confidence rises, consider $\left(1-\frac{n}{n+conf}\right)ek = \frac{conf}{n+conf}ek$. This term is then added to equation (30).

Putting all of this information together gives the following equation of w_i for the case of $t_i \le ek$.

$$w_i = \frac{n}{n+conf}t_i + \frac{cen+n+conf}{(2n+conf)(n+conf)}\sum_{j=i}^{n} t_j + \frac{conf}{n+conf}ek \tag{31}$$

To combine the two cases into one equation, let

$$weight_i = \begin{cases} n & \text{if} \quad t_i > ek \\ n+conf & \text{if} \quad t_i \le ek \end{cases} \tag{32}$$

and

$$S_i = \begin{cases} \sum_{j=i}^{n} t_j & \text{if} \quad t_i > ek \\ \sum_{j=i}^{n} t_j + ek\frac{conf(2n+conf)}{cen+n+conf} & \text{if} \quad t_i \le ek \end{cases} \tag{33}$$

then

$$w_i = \frac{n}{weight_i}t_i + \frac{cen+weight_i}{(n+weight_i)(n+conf)}S_i \tag{34}$$

In summary, the inverted sigmoid membership function for the i^{th} unit is defined by equation (25), where c is defined by equation (23), z_i is defined by equation (29), and w_i is defined by equation (34). In addition, this derivation is based on U=0.65 and L=0.001. However, these computations may appear complicated the method can be easily programmed for a computer. All computations are very quickly executed on a computer because this method is developed strictly in the case of very small and highly censored data sets.

In the next section, the data now expressed by membership functions are used to calculate a survival curve based on the concept of the Product-Limit estimator. With the Product-Limit estimator (Kaplan and Meier, 1958) the data can be viewed as having membership functions that are step functions from one to zero at the recorded time, whether they are censored or failure times. Then at each failure an "assessment" is made of the ratio that survived. In section 3.4 the estimator uses the membership functions as defined in this section and makes the "assessment" at all points between zero and *TOT*.

3.4 Fuzzy Product-Limit Estimator

An estimator of a survival curve is developed in this section that uses the fuzzy data and the concept of the Product-Limit Estimator (PLE). The PLE calculates the proportion of units that survived beyond time T_i to units available just before T_i, where T_i is the event of a failure. This proportion is calculated for each failure. This proportion of units available after a failure is conditioned on the proportion available at the previous failure, T_{i-1}, assessed at each failure. This method takes the view that the censored times between T_{i-1} and T_i are ignored, in the sense that these censored times provided information to the survival at time T_{i-1} but provide no additional information at time T_i. Thus, the PLE can make an "assessment" of survival at any given time; if a failure has occurred, then the estimate of survival is updated; otherwise it remains. The result is that a step function is created based on these assessments. Given enough time, the number of censored units approaches zero. In this case, at each point in the time axis x, the PLE curve can be calculated as the proportion of units surviving at time x. If this calculation is made at a censored time, the result is an underestimate.

The method proposed in this section uses the membership functions derived in the previous section to compensate for the underestimate resulting from the reassessment at each censored time. In addition, it addresses other problems associated with the PLE. It does address the problems with the PLE that occur when the last time(s) is(are) censored time(s). Namely, the estimate of the survival curve is greater than zero for all values on the time axis greater than t_n (the last failure time) and the estimated mean survival time is an underestimate. In practice, analysts often compensate for this problem by replacing the last censored time with a fail time slightly greater than the censored time. This modification to the PLE is a very limited case of using membership functions for each censored time as proposed in this chapter. This fuzzy PLE method also addresses problems that occur when very few failures are observed and the failure(s) are the last observed. In this situation, the mean survival is an overestimate. Consider Figure 3.7, these are the estimates of the survival curves generated from data set $A=\{(10,0), (50,1), (110,1), (240,1), (350,1)\}$ and $B=\{(10,1), (50,1), (110,1), (240,1), (350,0)\}$. The mean survival time estimated from the data in A is 10 and the mean survival time estimated using B is 350. The data sets can provide more information in both circumstances.

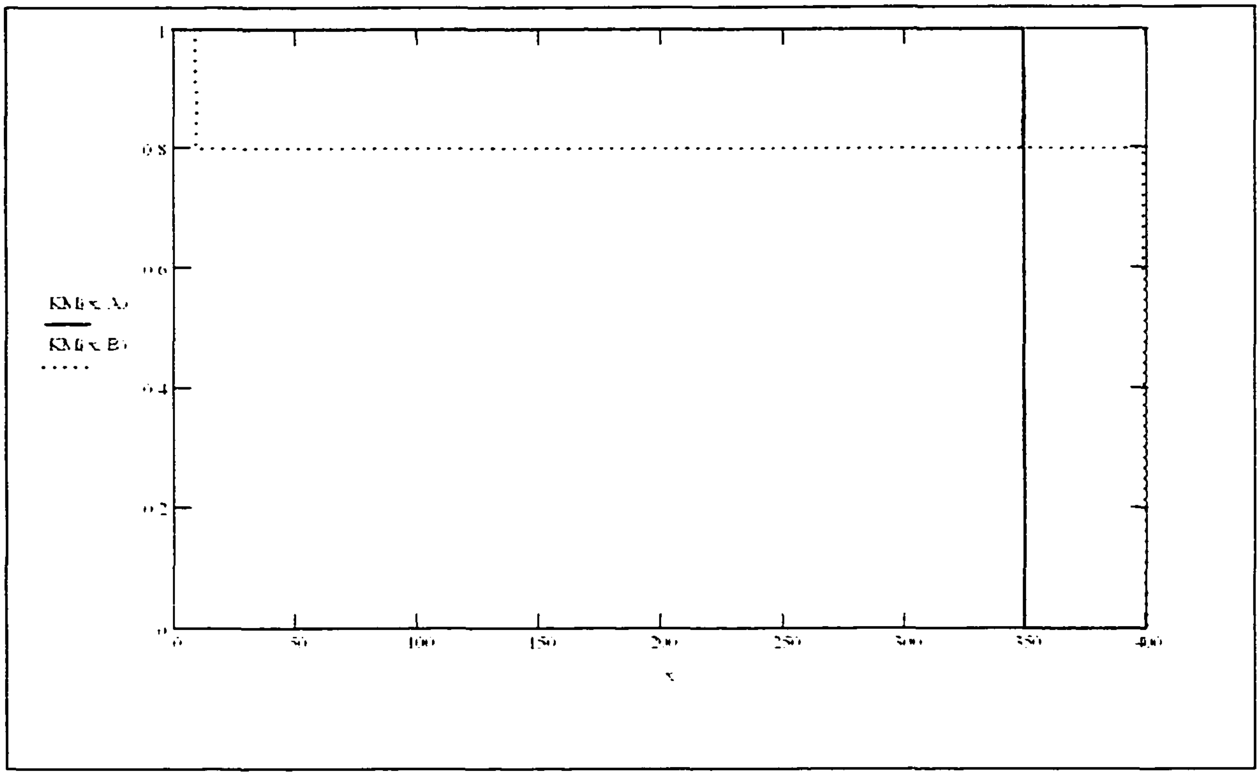


Figure 3.7 Kaplan-Meier Survival Curves

3.4.1 Fuzzy Product-Limit Estimator of the Survival Curve

Now, the data set expressed as membership functions are used to calculate an estimate of the survival curve. This estimator converges to the Product-Limit Estimator as the number of censored values approaches zero. At any given point x in the interval the percentage of units believed to be surviving can be calculated by adding the belief of the individual units at x and dividing by the total number of units, this is the proportion of surviving to total possible. This idea is directly related to the concept of the PLE since at each point x in the time axis the membership functions provide partial survival. In other words there is some belief of a failure (censoring has been supplanted by belief of survival) and the PLE makes an assessment at each failure.

Let *gl(x,t,i)* be the case that the belief is that the failure occurs immediately after the censored time (i.e., the belief is that a censored time is a failure time). The membership function for this case is written as

$$gl(x,t,i) = \begin{cases} \text{if} \quad t_i \text{ censored} & \\ \qquad 1 \quad \text{if} \quad x \le t_i & \\ \qquad 0 \quad \text{if} \quad x > t_i & \\ \text{otherwise} & \\ \qquad 1 \quad \text{if} \quad x < t_i & \\ \qquad 0 \quad \text{if} \quad x \ge t_i & \end{cases} \tag{35}$$

The subtle difference in the above inequalities is due to the fact that a censored unit is known to have survived until t_i, and an uncensored unit is known to have failed at time t_i. Let *gu(x,t,i)* be the case that the belief is that a censored unit survives past the last time in the data set. That is,

$$gu(x,t,i) = \begin{cases} \text{if} \quad t_i \text{ uncensored} & \\ \qquad 1 \quad \text{if} \quad x < t_i & \\ \qquad 0 \quad \text{if} \quad x \ge t_i & \\ 1 \quad \text{otherwise} & \end{cases} \tag{36}$$

Thus the functions *gl(x,t,i)*, and *gu(x,t,i)* represent the most pessimistic and optimistic belief of survival respectively as indicated by the data. As mentioned previously in this chapter these provide lower and upper bounds on the beliefs respectively. Given these beliefs in survival of the censored units the PLE for the pessimistic belief is calculated:

$$fl(x) = \frac{\sum_{i=1}^{n} gl(x,t,i)}{n} \tag{37}$$

and the PLE for the optimistic belief is calculated:

$$fu(x) = \frac{\sum_{i=1}^{n} gu(x,t,i)}{n} \tag{38}$$

Finally the fuzzy PLE indicates a partial belief at any given point and is calculated by

$$f(x) = \frac{\sum_{i=1}^{n} \mu_i(x)}{n} \tag{39}$$

The following example illustrates the value of the fuzzy PLE at a given point. The value of the fuzzy PLE can be calculated at any non-negative value in the time line and the result is represented graphically as a survival curve.

Example 3: Given the data {(10,1), (50,1), (110,0), (240,1), (350,1)}, at time 150 the belief of survival of the units are as follows: $t_1 = 0.626$, $t_2 = 0.865$, $t_3 = 0$, $t_4 = 1$, $t_5 = 1$. The value of the survival curve at this point is then calculated as

$f(150) = \frac{\sum_{i=1}^{5} \mu_i(150)}{5} = \frac{0.626 + 0.865 + 0 + 1 + 1}{5} = 0.6982$. It could be stated that there is a 0.374 + 0.135 = 0.509 amount of failure at time 150; that is, a failure in the amount of just over one half of a unit.

Survival curves based on the data from Example 3 are shown in Figure 3.8 for each of the three above beliefs.

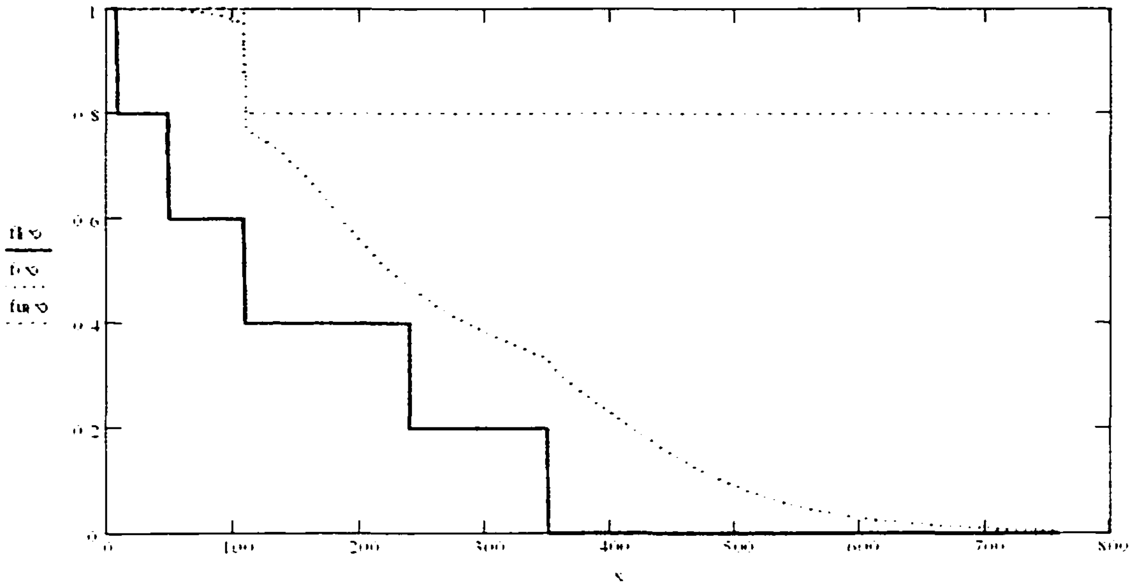


Figure 3.8 PLE Survival Curves for Pessimistic-*fl(x)*, Fuzzy-*f(x)* and Optimistic-*fu(x)* Beliefs

3.4.2 Mean Survival Time

The mean of the PLE is calculated as

$$\bar{S} = \sum_{i=1}^{n} s(T_{i-1})[T_i - T_{i-1}] \tag{40}$$

where $s(T_i)$ is the product limit estimate at time T_i and T_i is a failure time. Thus it is the area under the survival curve, or the integral of the survival curve provides an estimate of the mean survival time. So the mean survival time of the Fuzzy-PLE is calculated as

$$\tilde{S} = \int_0^{TOT} f(x)dx \tag{41}$$

With the data from Example 3, the PLE would estimate the mean survival time as 110. The integral of the pessimistic belief yields a mean survival time of 152, thus assuming that each censored time is a failure provides a larger estimate than the PLE. The Fuzzy-PLE gives a mean survival time of 270.370 in this example. The uncertainty in this estimate has a statistical component and a component associated with the fuzzy membership function. These issues are addressed in chapter 4.

3.5 Asymptotic Properties

In this section asymptotic properties of the Fuzzy-PLE are considered. This estimator uses several properties of the data set, all of them considered in this section. First, properties of w_i and z_i as defined by equations (34) and (29) respectively are taken into consideration. Then, properties of the membership function $\mu_i(x)$ as defined in equation (25) are explored.

3.5.1 User's Confidence

As the user's confidence *uc* approaches 100, *conf* approaches infinity. In this circumstance $z_i \rightarrow w_i$, which is an appropriate property as the limiting case of 100% confidence is approached. However, *conf* does not actually reach infinity since if *uc* = 100% there is no reason to use an estimator. Since $L = f(w_i), f(z_i) = U$ and $z_i \rightarrow w_i$, the result is an almost vertical graph from one to zero at w_i. If an approximation is made of the membership function with a straight line around the point w_i, the slope is $\frac{0.65-.0001}{w_i - z_i} = \frac{-64.99(f_i+1)}{TOT(100-uc)}$ for the circumstances used as a typical case in this development. This slope is steepest for censored values after the last failure and is dictated by the total operating time and the user's confidence in the opinion.

The effect on w_i as uc approaches 100 has two possibilities. The first case is that t_i is greater than ek and the second case is that t_i is less than or equal to ek.

Case 1 ($t_i > ek$): In this case $weight_i = n$ and $S_i = \sum_{j=i}^{n} t_j$ so that

$w_i = t_i + \dfrac{cen + n}{2n\ (n + conf)} \sum_{j=i}^{n} t_j \xrightarrow{conf \to \infty} t_i$. Therefore, the membership function approaches the step function $f_i(x) = \begin{cases} 1 & \text{if} \quad x \le t_i \\ 0 & \text{if} \quad x > t_i \end{cases}$ as uc approaches 100, the desired result for the asymptotic behavior.

Case 2 ($t_i \le gf$): In this case $weight_i = n{+}conf$ and

$$S_i = \sum_{j=i}^{n} t_j + gf \frac{conf(2n + conf)}{cen + n + conf} \text{ so that}$$

$$\lim_{conf \to \infty} w_i = \lim_{conf \to \infty} \left[\frac{n}{n + conf} t_i + \frac{cen + n + conf}{(2n + conf)\ (n + conf)} \sum_{j=i}^{n} t_j + \frac{conf}{n + conf} gf \right]$$

$$= \lim_{conf \to \infty} \frac{n}{n + conf} t_i + \lim_{conf \to \infty} \frac{cen + n + conf}{(2n + conf)\ (n + conf)} \sum_{j=i}^{n} t_j + \lim_{conf \to \infty} \frac{conf}{n + conf} gf$$

$$= \lim_{conf \to \infty} \frac{cen + n + conf}{(2n + conf)\ (n + conf)} \sum_{j=i}^{n} t_j + \lim_{conf \to \infty} \frac{conf}{n + conf} gf$$

Now applying L'Hopital's rule $\lim_{conf \to \infty} w_i = \lim_{conf \to \infty} \dfrac{1}{2conf + 3n} \sum_{j=i}^{n} t_j + \lim_{conf \to \infty} gf = gf$.

Therefore in this case the membership function approaches the step function $f_i(x) = \begin{cases} 1 & \text{if} \quad x \le gf \\ 0 & \text{if} \quad x > gf \end{cases}$ as uc approaches 100.

Now if *uc* approaches zero, *conf* approaches zero and the result is $w_i = t_i + \frac{cen+n}{2n^2}\sum_{j=i}^{n} t_j$ and $z_i = w_i + \frac{TOT}{f_i + 1}$. These are the functions developed in section 3.3. The asymptotic properties as *cen* and *n* approach the limiting values are considered next with *conf* held at zero.

3.5.2 Number of Observations and Proportion of Censored Values

Consider the case of approaching 100% censored data ($cen \to n$). In this case $z_i = w_i + TOT$, since $f_i = 0$ for $i = 1,\ldots,n$. As it should be, this is the most optimistic belief possible for z_i, depending on w_i. And, $w_i = t_i + \frac{1}{n}\sum_{j=i}^{n} t_j$ is the most optimistic belief possible for these membership functions.

Now, consider the case as *cen* approaches zero. Notice that *cen* must be at least one for this case to be useful. If *cen* is equal to zero, then no projection is made for the belief of survival. For the argument that follows assume that there is exactly one censored value and it has time t_i. So that $f_i = n-i$ where f_i is the number of failures that occur after time t_i thus, $z_i = w_i + \frac{TOT}{n-i+1}$. Then if the censored unit is the first data point (t_1), $z_1 = w_1 + TOT/n$ and if it is the last data point (t_n), $z_n = w_n + TOT$. When all evidence of failure is before t_n, z_n is extended *TOT* beyond w_n and with all evidence of failure after z_1 it is only extended to TOT/n beyond w_1. At first this value of z_1 may appear to be the opposite of the desired result but this is only half of the picture. Consider w_i for the case that the last time t_n is the censored value,

$w_n \approx t_n + \frac{1}{2n}\sum_{j=n}^{n} t_j = t_n\left(1+\frac{1}{2n}\right)$ then $z_n \approx t_n\left(1+\frac{1}{2n}\right)+TOT$. For the case that the first data point is censored, $w_1 \approx t_1 + \frac{1}{2n}TOT$ and $z_1 \approx t_1 + \frac{3}{2n}TOT$. Comparing the two situations reveals that in the case that the first data point is censored there is a 65% belief of survival of t_1 beyond censored time. For t_n the 65% belief of survival is very close ($1/n$) to t_n. Now, $z_n - t_n > z_1 - t_1$ and as demonstrated in section 3.3 this raises the belief before w_1 for the membership function $\mu_1(x)$.

Now consider the result when the number of units is n=1. This is the smallest possible case for n, since n=0 means there is no data to be analyzed. In this case $w_1 = t_1 + \frac{1+1}{2*1^2}t_1 = 2t_1$ and $z_1 = 2t_1 + t_1 = 3t_1$. Therefore, there is a 65% belief that the unit will survive twice as long and 0.01% belief it will survive three times as long as it has. Is this result reasonable? There is no evidence contrary to these beliefs in this situation. If there is some reason to think otherwise, for instance if the average operating temperature of a diesel engine has increased after the first year of operation, it may be indicative of wear out. This information can be incorporated in the form of expert knowledge, *ek*.

The next situation is purely an academic argument since the premise of this thesis revolves around very small data sets. Consider the case that the number of times n approaches infinity. This situation can occur two ways. In the first case, there is an unlimited number of units and the majority are censored, say a proportion p. Let $R=(1+p)/2$. Now in this case $w_i = t_i + \frac{R}{n}\sum_{j=i}^{n} t_j$.

Theorem 3.1: $w_i = t_i + \frac{1}{n}\sum_{j=i}^{n} t_j$ is bounded above by w_n, if $t_{i-1}\left(1+\frac{1}{n}\right) \le t_i$ for all i=2,...,n.

Proof: Consider $w_1 = t_1 + \frac{1}{n}\sum_{j=1}^{n} t_j = t_1\left(1+\frac{1}{n}\right) + \frac{1}{n}\sum_{j=2}^{n} t_j \le w_2 = t_2 + \frac{1}{n}\sum_{j=2}^{n} t_j$. The inequality is true since, by assumption $t_{i-1}\left(1+\frac{1}{n}\right) \le t_i$ for all i=2,...,n. Then by induction $w_2 \le w_3$, ..., $w_{n-1} \le w_n$, Thus by transitivity $w_i \le w_n$, i=1,...,n. #

In the current case $n \to \infty$ thus, $t_{i-1}\left(1+\frac{1}{n}\right) \to t_{i-1}$ and with the assumption that the times are ordered in an ascending fashion, $t_{i-1} \le t_i$. Therefore, by Theorem 3.1, $w_i = t_i + \frac{R}{n}\sum_{j=i}^{n} t_j$ is bounded by $w_n = t_n\left(1+\frac{1}{n}\right)$ and $\lim_{n\to\infty} w_n = t_n$. Finally, t_n is bounded above since nothing lasts for ever, especially equipment that contains moving parts. Whether t_n is a censored time or a fail time then it is a real number greater than zero. Hence w_i is bounded for i=1,..., n. In this case z_i grows without bound because it is assumed the number of failures is minimal, and the total number of times grows without bound. This circumstance is addressed in the next subsection.

The second case occurs in the scenario that there is a fixed number of units, say m, and as one fails its failure time is recorded and the unit is replaced; therefore, as n grows without bound the proportion of censored data approaches zero.

Theorem 3.2: $w_i = t_i + \frac{m+n}{2n^2}\sum_{j=i}^{n} t_j$ is bounded above by w_n if $t_{i-1}\left(1+\frac{m+n}{2n^2}\right) \le t_i$ for all i=2,..., n.

The proof directly follows the proof of Theorem 3.1 with $\left(1+\frac{1}{n}\right)$ replaced by $\left(1+\frac{m+n}{2n^2}\right)$. Following the argument used above, $t_{i-1}\left(1+\frac{m}{2n^2}+\frac{1}{2n}\right)\to t_{i-1}$ as $n\to\infty$ and by the assumption that the times are in ascending order, $t_{i-1}\le t_i$. Therefore, by Theorem 3.2, $w_i = t_i + \frac{m+n}{2n^2}\sum_{j=i}^{n} t_j$ is bounded by $w_n = t_n\left(1+\frac{m}{2n^2}+\frac{1}{2n}\right)$ and $\lim_{n\to\infty} w_n = t_n$. Therefore, w_i is bounded in this case as well.

Since *cen* << *n*, most of the times are failures and $f_i \approx n\text{-}cen$ (total number of failures). But suppose for instance that the last recorded time is a censored time. Then again the problem arises that $z_n = t_n + TOT$ grows without bound. This problem is addressed in the next subsection in the context of its effect on the membership function.

3.5.3 The Membership Function

For all cases on the time axis between zero and *TOT* the membership function is bounded above by one and below by zero. It is necessary only to consider censored values ($d_i=1$) since $\mu_i(x)$ is deterministic for failure times $d_i=0$. As $x\to t_i$ it is easily seen that $\mu_i(x)\to 1$. The membership function is dictated by the values of w_i, U, z_i, and L. Thus, the properties investigated above sufficiently explain the behavior of $\mu_i(x)$ in the limiting cases. The only situation remaining to be addressed is the problem of $z_i\to\infty$ as $n\to\infty$.

Now consider the problem of $z_i\to\infty$ that arose in the case of $n\to\infty$. This problem occurs because $z_i = w_i + TOT$ and TOT grows without bound as $n\to\infty$. With $\mu_i(x)$

the factor $\left(\frac{x-t_i}{TOT}\right) \le 1$ if $x - t_i \le TOT \Rightarrow x \le TOT + t_i$. This is why when considering the mean survival time the integral is evaluated from 0 to *TOT*. But in the current situation *TOT* grows without bound; therefore, this ratio approaches one. Next consider the factor $\frac{TOT}{z_i - t_i}$. The following relation holds. $0 < \left(\frac{TOT}{z_i - t_i}\right) = \left(\frac{TOT}{w_i + TOT - t_i}\right) \le 1$, since $w_i - t_i > 0$. Also, since $0<L<1$ then $0<1/L-1<\infty$ and in this development L=0.0001 thus $1/L$-1=9999.0. Likewise $1/U$-1= 0.53846. Next attention is turned to the exponents in $\mu_i(x)$.

The exponent $\frac{x-t_i}{z_i - t_i} \xrightarrow{n\to\infty} \frac{x-t_i}{w_i + TOT - t_i} \le 1$, since only $0 \le x < TOT+t_i$ are considered. And as x approaches *TOT* this ratio approaches one from the left. Thus the factor under this exponent approaches zero as x approaches t_i from the right and $\mu_i(x)$ approaches one. Finally, using equation (23) with L=0.0001 and U=0.65, the limit as $TOT \to\infty$ can be found, as follows:

$$c_i = \frac{(z_i - t_i)\ln\left(\frac{1-U}{U}\right) - (w_i - t_i)\ln\left(\frac{1-L}{L}\right)}{(z_i - t_i)\ln\left(\frac{w_i - t_i}{TOT}\right) + (w_i - t_i)\ln\left(\frac{TOT}{z_i - t_i}\right)} \tag{23}$$

$$= \frac{-0.6190(w_i + TOT - t_i) - 9.2102(w_i - t_i)}{(w_i + TOT - t_i)\ln\left(\frac{w_i - t_i}{TOT}\right) + (w_i - t_i)\ln\left(\frac{TOT}{w_i + TOT - t_i}\right)}$$

To consider the limit as *TOT* approaches infinity apply L'Hopital's rule 3 times to obtain

$$\lim_{TOT\to\infty} c_i = \lim_{TOT\to\infty} \frac{0.6190(w_i + TOT - t_i)}{(t_i - TOT - w_i)\ln\left(\frac{w_i - t_i}{TOT}\right) + TOT + 2(w_i - t_i)}$$

$$= \lim_{TOT\to\infty} \frac{0.6190 * TOT}{-TOT\ln\left(\frac{w_i - t_i}{TOT}\right) + 2TOT + w_i - t_i}$$

$$= \lim_{TOT\to\infty} \frac{0.6190}{3 - \ln\left(\frac{t_i - w_i}{TOT}\right)}$$

$$= 0.$$

In addition, applying L'Hopital's rule yields $\lim_{TOT\to\infty} c_i \frac{x - t_i}{z_i - t_i} = 0$.

Now putting all of these parts together, it can be verified that

$$\lim_{TOT\to\infty} \mu_i(x) = \begin{cases} 0 & \text{if } x \to TOT \\ 1 & \text{if } \quad x \to t_i \end{cases}.$$

Therefore as z_i approaches infinity the membership function behaves as desired. Actually, since L=0.0001 (a value close to zero) is considered in this presentation the membership function does not go to zero as x $\to TOT$. Instead it approaches $\frac{1}{1+9999} = 0.0001$. With this last examination all asymptotic properties of this estimator have been considered.

3.6 Chapter Summary

In this chapter a method is developed that uses fuzzy set theory to compensate for problems that occur due to small data sets that are highly censored. For each censored value a fuzzy membership function is used to describe the belief of survivability of the associated unit. This membership function is data generated, in that characteristics, such as the size of the data set, proportion censored, the magnitude of values, and total operating time, are used to define the shape of the membership function. The membership function for a failed time is a step function equal to one from time zero to the recorded time of failure and zero from the fail time and beyond.

In addition, the method allows for the input of an "expert opinion" of the survival time and an associated confidence level of the expert opinion. It is suggested that this "expert opinion" may come from several sources such as an estimate taken from like data, an "expert" familiar with the equipment under study, from the same equipment used under a different mode of operation, or data that had been poorly collected previously. Given this information, the membership functions generated from the data are adjusted in accordance with this input.

The data set, now in terms of membership functions, is used to estimate a survival curve. This estimator converges to the Product-Limit estimator (PLE) (Kaplan and Meier, 1958) when the number of censored times is zero. This estimator is based on the concept of the PLE. Taking the perspective of the PLE that it starts at time zero, assesses the data for survivability at each point on the time axis as time increases and only makes a new calculation in the presence of a failure; otherwise, the previous estimate holds. From this view, the "Fuzzy-PLE" is the same estimator as the PLE. The difference is in the data. With the Fuzzy-PLE developed in this chapter, the data are in the form of membership functions that represent a belief in the survivability of the equipment. If this belief is less than one, then there is some belief of failure. Thus at any point x in the time axis, if any membership function $\mu_i(x)$, $i=1,\ldots, n$ exists such that $0 < \mu_i(x) < 1$, then there is a belief that some failure may exist. As with the PLE the Fuzzy-PLE recalculates the survivability in the presence of the failure.

Finally, asymptotic properties of the membership functions are presented. It is shown that the membership functions behave appropriately at the limiting values of the number of censored units, the user's confidence in the expert opinion, and the number of observed times.

CHAPTER 4

UNCERTAINTY ESTIMATES IN THE FUZZY PRODUCT-LIMIT ESTIMATOR

4.1 Introduction

In Chapter 3 an estimator of the mean survival time is developed. The estimator is used with data sets containing censored values. Taking the view that a censored time is a vague failure time, the censored values are then represented by fuzzy membership functions that describe a belief of continued survival of the associated unit. Associated with any estimate is uncertainty. Here two distinct types of uncertainty are quantified in the estimate. The uncertainty due to the randomness in the recorded times and vague uncertainty in the failure of the censored units. The current chapter addresses the problem of providing some confidence bounds around the estimate obtained this way. The method uses the bootstrap (Efron, 1979) as a basis for obtaining a confidence interval of the estimate developed in Chapter 3. The new method provides confidence intervals that quantify statistical uncertainty as well as the vague uncertainty in the censored times.

The bootstrap method (Efron, 1979) as described in Chapter 2, is a computational method for obtaining standard errors of a statistic. The method can also be used to estimate bias and confidence intervals for any statistic. This method works well and is intended for small data sets. In the current study, it is desired not only to quantify the statistical uncertainty but also the uncertainty quantified by the fuzzy membership functions. Incorporating beliefs in the form of fuzzy membership functions into the estimation of survival curves has quantified vague uncertainty in the data, since an inference is made about the continued survival of a censored unit.

In Chapter 3 the Fuzzy-PLE is introduced to overcome some of the problems of the PLE and results in relatively smooth survival curves that described the information from small and highly censored data sets. The survival curves are then used to produce an estimate of the mean survival time. The intention in the current chapter is to quantify statistical uncertainty and uncertainty quantified by the fuzzy sets in the estimation of the survival time.

4.1.1 Efron's Bootstrap Algorithm

The bootstrap algorithm is a computational way of obtaining an approximation to the numerical value of the standard error of an estimator. Figure 4.1 provides the procedure for calculating the standard error se_B of an estimator $s(x)$.

1. Take B bootstrap samples $x^{*1}, x^{*2}, \ldots, x^{*B}$, each consisting of n data values drawn with replacement from x.
2. Evaluate the bootstrap replication corresponding to each bootstrap sample,

   $\hat{\theta}^*(b) = s(\mathbf{x}^{*b}), \quad b = 1,2,\ldots,\mathrm{B}$

3. Estimate the standard error $se_F(\hat{\theta})$ by the sample standard deviation of the B replications (Efron and Tibshirani, 1993):

$$se_B = \left\{ \frac{1}{B-1} \sum_{b=1}^{B} \left(\hat{\theta}^*(b) - \bar{\hat{\theta}}^* \right)^2 \right\}^{\frac{1}{2}} \text{ where } \quad \bar{\hat{\theta}}^* = \frac{1}{B} \sum_{b=1}^{B} \hat{\theta}^*(b)$$

Figure 4.1 The Bootstrap Algorithm

In general, the Bootstrap method can be used on fuzzy data to get an estimate of the standard error and confidence intervals (Hung, 2001). This method can be used directly to estimate the standard error of the mean survival time calculated from the Fuzzy-PLE, resulting in a point estimate with an associated error. One issue that needs to be addressed is when selecting a bootstrap sample, whether each data point is taken as an ordered pair (t_i, d_i) or is t_i randomized and then the censor status d_i randomized separately. In the current dissertation it is desired to get an estimate of how much uncertainty is due to the data and how much is due to the censored values; thus each data point is taken as an ordered pair.

Several methods are investigated for quantifying the two types of uncertainties. What does this mean? A confidence interval is produced with perhaps, a 90% coverage probability for the mean survival time, but this estimate is based on an estimator that uses fuzzy sets to quantify a belief in continued survival of a censored unit. An estimate is needed that describes the amount of vague uncertainty used in the estimate and in calculating the confidence interval. This vague uncertainty is essentially the uncertainty quantified by the fuzzy membership functions in the data and therefore, the confidence interval (a measure of statistical error) needs to include a description of the vague uncertainty in the estimator.

The percentile method of confidence intervals is a bootstrap method that uses percentiles of the bootstrap distribution to generate a confidence interval. This method is the same as is used with standard normal confidence intervals, except that the standard normal method uses percentiles from a normal distribution. After taking the bootstrap replications, the endpoints of the confidence interval are the $100\alpha^{th}$ percentile and the $100(1-\alpha)^{th}$ percentile of the bootstrap distribution. For instance, if B=2000 bootstrap replications are generated and a 90% coverage probability (α=0.05) is desired then the 100^{th} ordered value is the lower endpoint and the 1900^{th} ordered value is the upper endpoint.

Confidence intervals based on bootstrap percentiles are not necessarily symmetric. They reflect the shape of the data. This property gives better coverage probability than confidence intervals based on the standard normal percentiles and is referred to as transformation-respecting, in reference to transformations that are made when the assumption of a normal distribution needed for standard normal theory are violated. The Bias Corrected and Accelerated (BC_a) method discussed in Chapter 2 is an improvement to the percentile method that takes into account bias in the estimator and deviations from the assumption needed in standard normal approximation theory that the standard error in the estimator is the same for all parameter values.

In the following subsections several options are developed that quantify the uncertainty in the estimator. Confidence intervals are estimated using the BC_a method and the "vague uncertainty" is quantified in several ways using data generated from the bootstrap samples.

4.2 Vague Uncertainty in the Fuzzy-Product Limit Estimator

How should the amount of vague uncertainty in the estimator be quantified? One straightforward answer is to consider the difference between the estimate of the mean survival time calculated using Fuzzy-PLE and the estimate of the mean survival time calculated using the most pessimistic view of the data. In essence, this is the amount of vague uncertainty quantified using the fuzzy membership functions aggregated across the Fuzzy-PLE. Mathematically, recall in Section 3.4.2 the mean survival time of the Fuzzy-PLE is calculated as $\widetilde{S} = \int_0^{TOT} f(x)dx$. Now, let the mean survival time of the pessimistic curve be $S_{pes} = \int_0^{TOT} fl(x)\,dx$. The amount of vague uncertainty is then quantified as

$$S = \widetilde{S} - S_{pes.} \qquad (42)$$

Visually, this result can be described as the area between the Fuzzy-PLE of the survival curve and the estimate of the survival curve taken using the pessimistic view of the data. Figure 4.2 shows the graphs using the data from Example 3, {(10,1), (50,1), (110,0), (240,1), (350,1)}. In this example, $\tilde{S} = 270.370$, $S_{pes} = 152.0$, and thus $S = 118.370$.

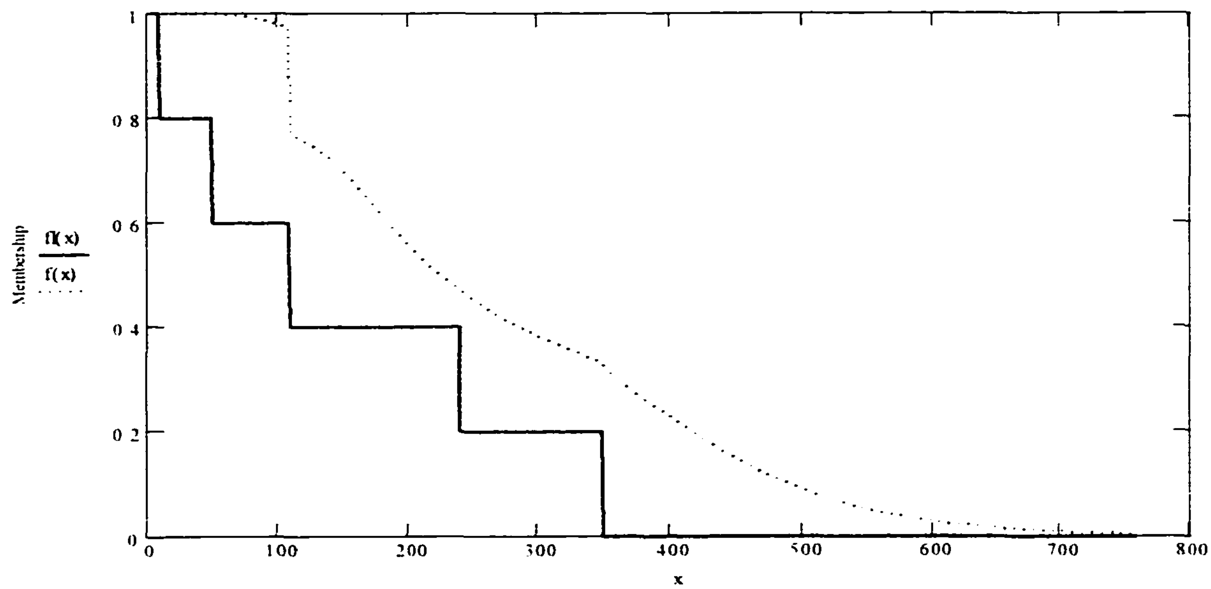


Figure 4.2 PLE Survival Curves Using the Pessimistic and Fuzzy-PLE

This method of quantifying the vague uncertainty is easily described and calculated. One would then report the mean survival time $\tilde{S}$, standard error ($se_B(\tilde{S})$) calculated using the bootstrap, the lower and upper bound of the confidence interval, and the vague uncertainty S. Thus, for the above data set with B = 1200 then $\tilde{S} = 270.370$, $se_B(\tilde{S}) = 9.704$, the 90% confidence interval $CI(\tilde{S}) = (99.516, 492.064)$, and $S = 118.370$ are reported. This method gives the point estimate $\tilde{S}$ for the mean survival time, a confidence interval and some indication of the amount of vague uncertainty quantified in the estimator.

4.2.1 Relative Vague Uncertainty

The vague uncertainty described above is an absolute measure of the uncertainty and therefore problem specific. For example, to report that the vague uncertainty is S = 5.0 has a different meaning if the estimate of mean survival time is $\widetilde{S}$ = 200.0 than if the estimate of mean survival time is $\widetilde{S}$ = 6.0. In the first case there is very little vague uncertainty relative to the estimate, and in the second case most of the estimate is the vague uncertainty. For a given problem it is better to have a small amount of vague uncertainty, but this alternative is dictated by the amount of information available.

Another measure that provides information about the vague uncertainty that can be compared between samples is the relative uncertainty. The relative uncertainty is measured as the amount of uncertainty quantified using S relative to the estimate of the mean survival time.

$$S_R = S / \widetilde{S} \tag{43}$$

For the above problem the relative vague uncertainty is then given as S_R = 118.370/270.370 = 0.438. This result says that, in this case about 44% of the estimate is based on the belief of continued survival as quantified by the fuzzy membership functions.

4.2.2 Mean Vague Uncertainty in the Empirical Distribution

An extension to the previous method is to calculate S for each of the B bootstrap samples, called $S^{*b} = \widetilde{S}^{*b} - S_{pe}^{*b}$ for the b^{th} bootstrap sample. Then take the mean of the B, S^{*b} as the vague uncertainty in the estimate.

$$\bar{S} = \frac{1}{B}\sum_{b=1}^{B}\left(\widetilde{S}^{*b} - S_{pes}^{*b}\right) \tag{44}$$

This method provides an estimate of the vague uncertainty for the Fuzzy-PLE in the empirical distribution of the data. Thus, a point estimate $\widetilde{S}$ with associated confidence interval can be reported along with the mean of the vague uncertainty in calculating the bootstrap confidence interval. For the above data set with B = 1200, $\overline{S}$ = 121.022 is obtained. This result indicates that, for the given data, the empirical distribution of the data has more vague uncertainty than is indicated by S itself.

4.2.3 Confidence Interval for the Vague Uncertainty in the Estimate

Taking the previous method one step further, S can be used as the estimate of the vague uncertainty and calculate the bootstrap confidence interval for S as well as $\widetilde{S}$ from the same bootstrap replications. The point estimate with confidence interval can be reported along with an estimate of the vague uncertainty and a confidence interval about the vague uncertainty. The vague uncertainty is calculated for each bootstrap sample. Since the amount of vague uncertainty is known for each bootstrap sample this information can be used to find a confidence interval about S using the BC_a method.

Now an estimate of the mean survival time with a confidence interval is reported. In addition, an estimate of the vague uncertainty with confidence interval is reported. Stated another way, an estimate of survival time with random uncertainty is given along with an estimate of the vague uncertainty in the estimate as well as the random uncertainty in the estimate of the vague uncertainty.

In the above data set, {(10,1), (50,1), (110,0), (240,1), (350,1)} with B = 1200 a 90% confidence interval for both estimates is obtained resulting in the following values: $\widetilde{S}$ = 270.370, CI($\widetilde{S}$) = (99.515, 492.064), S = 118.370, and CI(S) = (30.450, 245.854).

The methods discussed so far for describing the vague uncertainty in the estimate are relatively easy to calculate. Bootstrap samples are taken, the empirical distribution of both $\tilde{S}$ and S are obtained, and then the BC_a method is used to obtain confidence intervals for each of the estimates. The uncertainties are then reported in terms of point estimates and confidence intervals. The next method describes the two types of uncertainties in the estimate visually using the concept that the uncertainties are two distinct dimensions represented in a Cartesian coordinate system.

4.2.4 Two Dimensions of Uncertainty

Two distinct types of uncertainty are being quantified in the estimate. The random uncertainty is a statistical error due to the randomness in the data. The second type of uncertainty being quantified is the vague uncertainty caused by the lack of knowledge of failure time in the censored units. Using this perspective, the uncertainty can be described graphically in the Cartesian coordinate system by a confidence interval on the abscissa describing the random uncertainty and a bar graph in the ordinate direction describing the vague uncertainty.

In this method, the most pessimistic belief of survival, $fl(x)$ is calculated and the mean survival time of this curve, S_{pes} is generated. Then the mean survival time calculated from the Fuzzy PLE, $\tilde{S}$ is generated. And, as described previously, the difference of these two values $S = \tilde{S} - S_{pes}$ represents the amount of vague uncertainty that is quantified in the estimate. That is, the extrapolation that occurred from using fuzzy sets as opposed to assuming that the belief of survival is zero after censoring time. This difference is calculated for each of the B bootstrap samples. Then a confidence interval is calculated for $\widetilde{S}$ using a bootstrap method. The result is a point estimate and confidence interval for $\widetilde{S}$. In addition, S has been generated for each of the bootstrap samples resulting in an empirical distribution of the vague uncertainty in the estimates. Thus at any point in the confidence interval the amount of vague uncertainty at that point may be reported. Or graphically, a graph of the distribution of the vague uncertainty over the confidence interval may be reported. This method provides a more complete description of the vague uncertainty across the confidence interval because there exist patterns not seen when just considering the mean or confidence interval about S. It is important to note that the amount of vague uncertainty is not stochastic. It is deterministic, given the data set. The data set contains the statistical error.

Computer programs have been developed that calculate the estimates and create the graph of the uncertainties. Considering the data set {(10,1), (50,1), (110,0), (240,1), (350,1)} with B = 1200 a screen shot depicting the graph is shown in Figure 4.3. The "x" represents the mean survival time and the vertical lines that extend from the bottom to the top of the graph represent the lower (99.5156) and upper (492.0641) bounds of the confidence interval. Each vertical line represents the amount of vague uncertainty calculated using S^{*b}, at that point on the abscissa in the confidence interval.

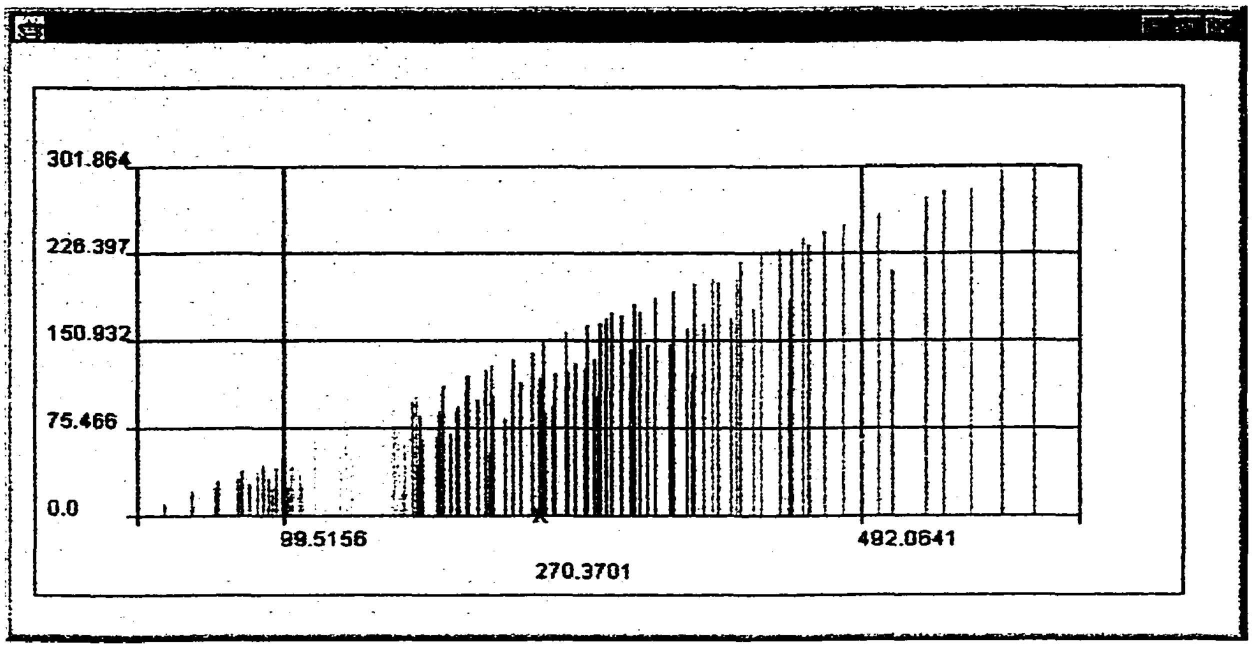


Figure 4.3 Graph of Random and Vague Uncertainties

In this example, a pattern is recognized in the vague uncertainties. An increase in the amount of vague uncertainty goes from left to right across the confidence interval. Since the censored values are evenly distributed in the data set (the only failure is the middle value) bootstrap replications that produce large values are obtained with large censored values resulting in much vague uncertainty in the estimate. Figure 4.4 shows the screen shot of the graph of uncertainties for the data set {(10,1), (50,1), (110,0), (240,0), (350,0), (410,0), (450,0)}with B = 6000. In this example only the early times are censored so that the bootstrap replications that produce large values are obtained with large failure times and do not necessarily contain more vague information than the bootstrap replications that produce the small values.

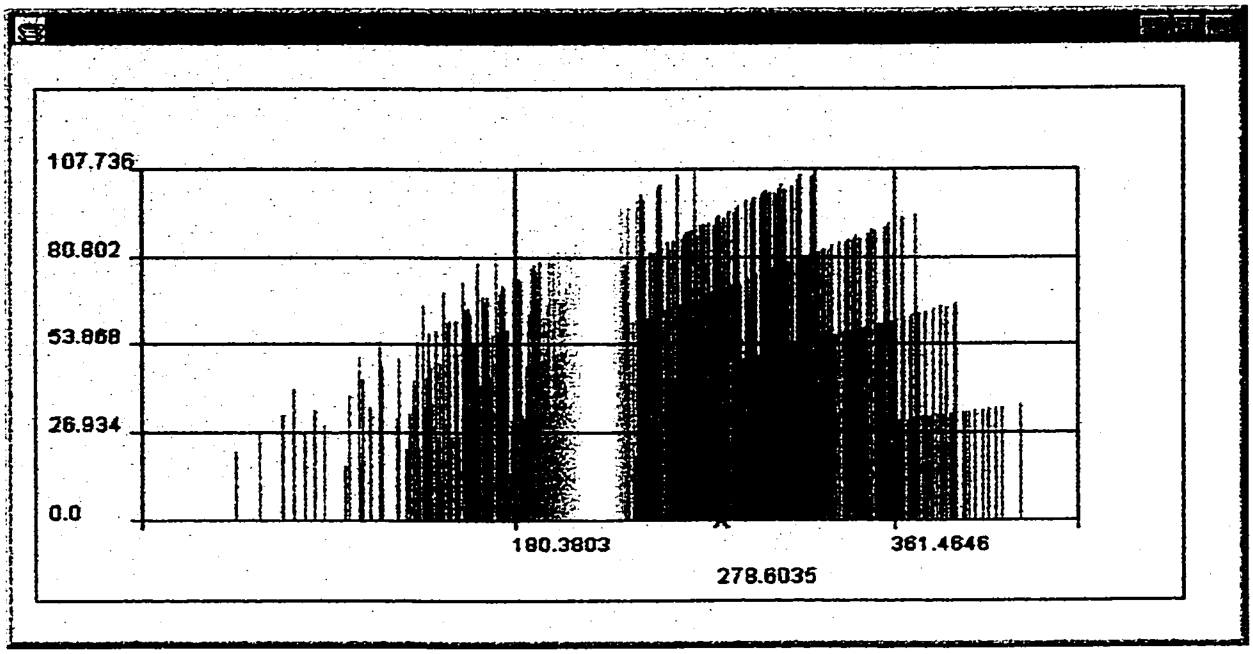


Figure 4.4 Random and Vague Uncertainties with Few Early Censored Values

Considering the uncertainty in this visual form provides a description of the random and vague uncertainties. In addition, a description of the interaction of vague uncertainty across the confidence interval is provided.

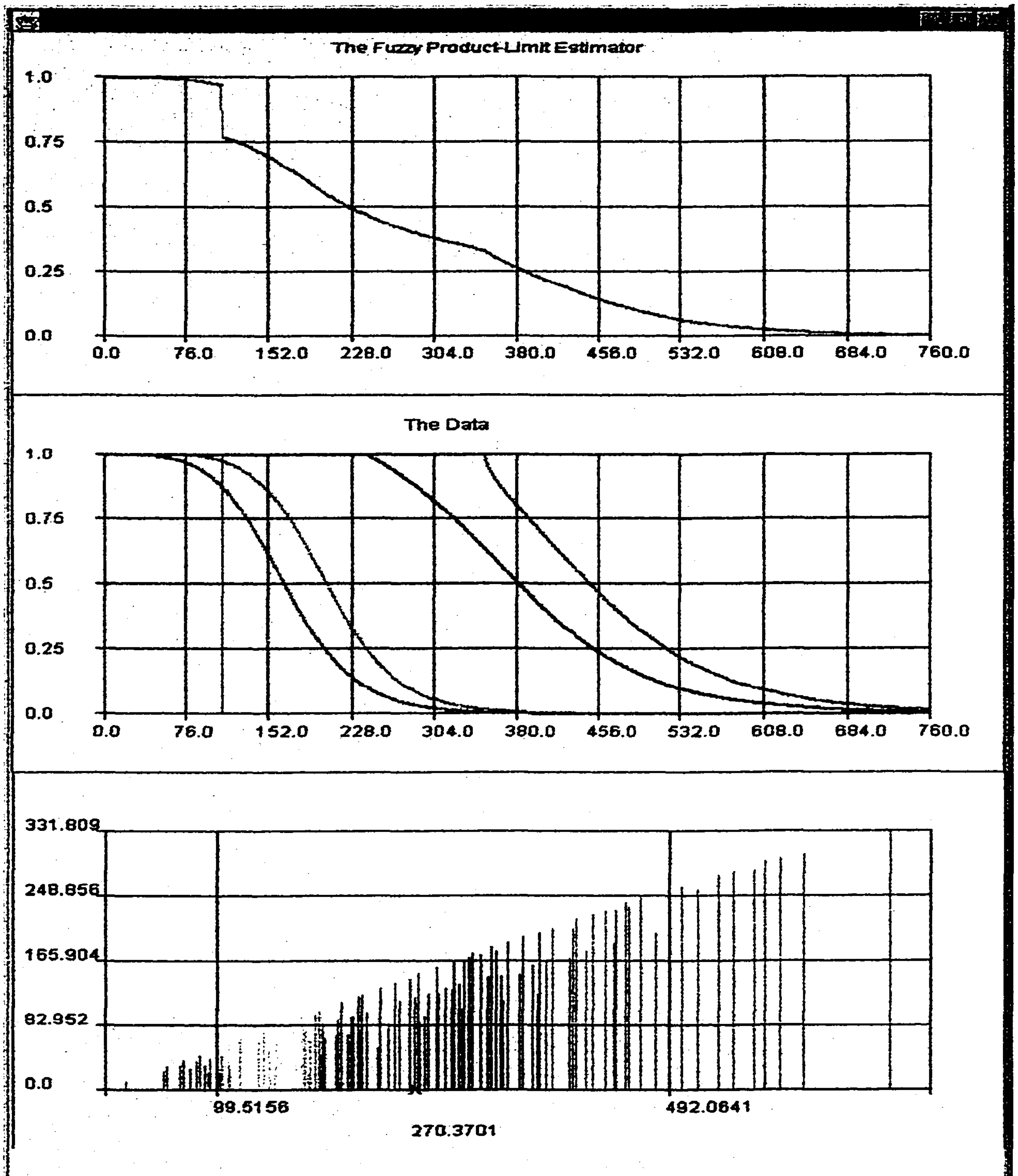
4.3 Chapter Summary

The methods described in this chapter are used to quantify and describe the uncertainty in the estimator. The uncertainty has random and vague components that represent different phenomena. The random uncertainty is from the randomness of the data and the vague uncertainty is from of the censoring of the units resulting in vague failure times.

Efron's bootstrap methods are used to obtain an empirical distribution of the data and the BC_a method is used to obtain a confidence interval about the mean survival time estimated using the Fuzzy-PLE. Because the Fuzzy-PLE uses fuzzy sets to quantify the uncertainty in the censored values, some measure of this "vague uncertainty" is needed. Several measures of the vague uncertainty are developed in this chapter. First S is considered and provides a straightforward measure of the amount of vague uncertainty actually used in the estimate. Secondly, the mean of the vague uncertainty $\overline{S}$, estimates the average vague uncertainty in the empirical distribution from the bootstrap replications. Since S is an estimate of the vague uncertainty a confidence interval, CI(S), of S is considered. Finally, since it recognized that the amount of vague uncertainty changes across the confidence interval, a graph of this uncertainty is also presented as an analytical tool.

The final analysis should consider all of these measures to describe the uncertainty in the estimate. Figure 4.5 shows all of the information of an estimate for the data set {(10,1), (50,1), (110,0), (240,1), (350,1)} as described. The top graph is the survival curve as estimated with the Fuzzy Product-Limit Estimator. The middle graph is the data set represented by fuzzy membership functions. The bottom graph depicts the uncertainties in the estimate. Note that the top two graphs are on the same scale and the bottom graph is on a different scale. Finally, a table is shown with the estimates and associated confidence intervals.

The next chapter tests the performance of the Fuzzy Product-Limit Estimator in estimating the mean survival time. The tests are run repeatedly for several distributions and comparisons are made with the performance of the Product-Limit Estimator.



$\tilde{S}$	$\mathrm{CI}(\tilde{S})$	$\mathrm{se}_B(\tilde{S})$	$\overline{S}$	S	$\mathrm{CI}(S)$
270.370	(99.516, 492.064)	9.704	121.022	118.370	(30.450, 245.854)

Figure 4.5 Estimate and Uncertainty Measures

CHAPTER 5

SIMULATIONS AND EMPIRICAL RESULTS

5.1 Introduction

This chapter tests the performance of the Fuzzy PLE. Testing the estimator is accomplished by generating random data from several distributions consisting of censored and failure times. Next, estimates are made using the Fuzzy-PLE with associated uncertainty measures. The tests are run repeatedly for each distribution, the data analyzed, and comparisons are made with the actual mean of the distribution and the performance of the PLE on each data set. Several issues are addressed with these simulations. It is demonstrated that when the sample is taken early relative to the mean of the distribution that the estimate of the mean survival time obtained using the Fuzzy-PLE is superior to the estimate obtained using the PLE. And that with matured data the estimate obtained using the Fuzzy-PLE agrees with that of the PLE.

In this chapter all of the estimates are made with the parameters *ek* and *uc* set to zero. These values indicate that no expert knowledge is considered in making the estimates. Also the parameter U is set to 0.65. This value of U is a moderate value of U that is used in the development of the Fuzzy-PLE in chapter three. Varying these parameters is then considered separately in the next chapter.

5.2 Empirical Testing of the Fuzzy Product-Limit Estimator

It is desired to demonstrate the efficacy of the Fuzzy Product-Limit Estimator using data generated from several probability distributions. To test and make comparisons with the Product-Limit Estimator ten rows of data are generated from the given distribution. Each row represents a unit of the equipment for which the survival time is being estimated.

5.2.1 Generation of Data Sets

In this chapter the estimates will be made at time twenty-four for all of the distributions. Thus for a given row (piece of equipment) if a time less than 24 is generated it will be considered a failure time and another number (failure time) is randomly generated from the distribution for the unit. This process is repeated until the sum of the row is greater than or equal to twenty-four. Each row then represents the failures of the given unit from time zero to twenty-four and if the sum exceeds twenty-four, then the last time in the row is truncated and represents a censored value. This procedure is repeated for each of the ten rows and all of the failure times and censored times form the data set to be analyzed at time twenty-four.

Table 5.1 shows data generated from an *exponential*(24) distribution and Table 5.2 gives the data set containing the information known at time twenty-for based on the data in Table 5.1. The first column in Table 5.1 is the unit number of a piece of equipment. The second column of all zeros is the starting time. For example, the second row in Table 5.1 contains three times that sum to 39.744. Because the first two values sum to 23.098, the last value of 16.646 is truncated to 0.902, which is treated as a censored time. Now the first two values summed with the censored value equals exactly twenty-four. This is the information known at time twenty-four for the second unit and is what is used in the analysis. Thus the failure and censoring times for the second unit can be found in Table 5.2 at rows two, three and four. These calculations are performed for each of the ten rows of data in Table 5.1.

Table 5.1 *exponential*(24) Raw Data

Unit	Start	Time 1	Time 2	Time 3
1	0	34.982	-	-
2	0	14.106	8.992	16.646
3	0	17.398	11.471	-
4	0	3.556	51.019	-
5	0	18.498	1.051	34.702
6	0	40.681	-	-
7	0	26.848	-	-
8	0	14.113	22.662	-
9	0	27.828	-	-
10	0	1.617	6.676	18.452

In this example, from the ten units, the nineteen data points shown in Table 5.2 are generated, ten are censored values and nine represent the times of failure. The first column represents the unit to which the time pertains. The second column (t_i) is the time and the third column (d_i) provides the censoring indicator. The entire data set represents a snap shot in time of the information known at time twenty-four in the lifetime of these units.

Table 5.2 Censored and Failure Times Taken at time Twenty-Four

Unit	t_i	d_i
1	24.000	1.0
2	14.106	0.0
2	8.992	0.0
2	0.902	1.0
3	17.398	0.0
3	6.602	1.0
4	3.556	0.0
4	20.444	1.0
5	18.498	0.0
5	1.051	0.0
5	4.451	1.0
6	24.000	1.0
7	24.000	1.0
8	14.113	0.0
8	9.887	1.0
9	24.000	1.0
10	1.617	0.0
10	6.676	0.0
10	15.707	1.0

Then all of the values, censoring, and failure times are arranged into ascending order with their respective censoring indicator. The knowledge of which unit produced a particular time is not important in the estimates and is therefore omitted. Table 5.3 shows the data then used to make the estimates. This is the form of the data sets used throughout the remainder of this dissertation.

Table 5.3 Ordered Data from Table 5.2

t_i	d_i
0.902	1.0
1.051	0.0
1.617	0.0
3.556	0.0
4.451	1.0
6.602	1.0
6.676	0.0
8.992	0.0
9.887	1.0
14.106	0.0
14.113	0.0
15.707	1.0
17.398	0.0
18.498	0.0
20.444	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

5.3 Simulations and Estimates from Several Distributions

As stated above, the estimate is made at time twenty-four for all the tests. The difference is in the distributions. Simulations will be performed on data from the *exponential*(48), *exponential*(24), and *exponential*(6) distributions. By taking all estimates at time twenty-four using the method of simulating data as stated in section 5.2, it is seen that the data set from the *exponential*(6) distribution is matured with only 15% to 31% censored values. Both estimates made using the Fuzzy-PLE and PLE are relatively accurate with this data. The *exponential*(24) distribution is considered as a middle case for comparative purposes. The data sets from this distribution contain between 32% to 71% censored values which is a relatively high percentage for making estimates using the PLE. The data from the *exponential*(48) distribution represents the other extreme since the data have not had time to mature. The estimate is made early and there is a large proportion (47% - 100%) of censored values.

Several runs are made for each of the distributions. The number of runs differs for each of the distributions. For instance the estimates made from the *exponential*(6) are rather consistent across samples since the data are matured at time twenty-four. The data are considered matured in the sense that for the exponential distribution most values occur before the mean. So for the *exponential*(6) distribution, at time twenty-four many failures have been observed. In contrast, a time of twenty-four is very early to be making estimates for the *exponential*(48) distribution. The result is a large variance in the estimates made with small highly censored samples from the distribution. Thus fewer runs are needed with the *exponential*(6) than with the *exponential*(48) data to obtain the same statistical significance in the analysis of the results. For each data set the following information is collected:

n - size of the data set

$\tilde{S}$ - mean survival time estimated using the Fuzzy-PLE

S_{pes} - mean survival time estimated using the pessimistic view of the data

P- mean survival time calculated using the Product-Limit Estimate

$se_B(\tilde{S})$ - standard error of $\tilde{S}$ calculated using the bootstrap method

$l\tilde{S}$ - lower bound on the confidence interval for $\tilde{S}$

$u\tilde{S}$ - upper bound on the confidence interval for $\tilde{S}$

S - vagueness in the estimate of $\tilde{S}$

$\bar{S}$ - mean of the bootstrap replications of S

$se_B(S)$ - standard error of S calculated using the bootstrap method

lS – lower bound on the confidence interval for S

uS – upper bound on the confidence interval for S.

In evaluating the efficacy of the Fuzzy-PLE several items are of interest here. The most important is whether or not the Fuzzy-PLE can be used to accurately estimate the mean of the distribution. The answer to this question is found by observing how close the estimate is to the actual mean of the distribution and whether the confidence interval covers the mean of the distribution or not. Secondly, a comparison of the performance of the mean survival time calculated using the Fuzzy-PLE relative to the mean survival time calculated using the PLE is of interest since the motivation and development of the Fuzzy-PLE is intended to overcome some of the problems that occur with the PLE under the circumstances addressed in this dissertation. The next three subsections consider the simulations from the three respective distributions.

5.3.1 Simulations with the *exponential*(48) Distribution

For the *exponential*(48) distribution one hundred fifty data sets are randomly generated. All of the data sets, in the form like that of Table 5.3 are provided in Appendix A. In these simulations it is known that the population parameter being estimated is 48. One of the data sets (data set 78 in Appendix A) contain only the ten values censored at time 24 and two data sets (data sets 11 and 65 in Appendix A) contain a maximum of twenty-one (eleven failures and ten censored) values Table 5.4 and Table 5.7 show these respective data sets.

5.3.1.1 Minimum Number of Failures. Consider the data set consisting of 10 values all censored at time 24. This is the extreme case in which no failures are observed. This case occurred once in the random generation of the one-hundred fifty *exponential*(48) data sets. With this case the Product-Limit Estimator results in no answer since no failures are observed and a failure is required to make an estimate with this statistic (actually an answer of 0 is returned from the computer program when no failures occur). The pessimistic view of the data (assume censored times are failure times) results in an estimate of 24, and the estimate of mean survival time made using the Fuzzy-PLE results in 55.802. The bootstrap confidence interval results in the value of 55.802 for both the upper and lower bound since all of the data is the same. Thus the only measure of vagueness is $S = 31.802$.

Table 5.4 Ten Data Points Censored at Time 24

t_i	d_i
24.0	1.0
24.0	1.0
24.0	1.0
24.0	1.0
24.0	1.0
24.0	1.0
24.0	1.0
24.0	1.0
24.0	1.0
24.0	1.0

Therefore, under this circumstance all units have the same time in service and no failures are observed. From an intuitive perspective ten datum are available, all containing censored values of 24. With no other information it is estimated that the units will last more than twice as long than has been observed (55.802), with the understanding that the amount of vagueness in the estimate is 31.802. Does this estimate seem to be an overestimate? Consider the probability of observing ten failures greater than or equal to 24 from a general *exponential*(λ) distribution. The observation time and the pessimistic estimate of the mean survival time are both 24. If $\lambda = 24$ then the probability of randomly generating ten values greater than or equal to 24 is 0.00005 or 1 in 20,000, a rare event. For larger values of the mean the probability increases. If the mean is actually 55.802 then the probability of this event is 0.0136. It requires a mean of 150 to have a probability of 0.2019 (i.e. a 1 in 5 chance) and a mean of 350 to get a probability of 0.5037. These probabilities are based on observing ten independent failures greater than or equal to twenty-four, what is observed is ten censored values, which means the failures have not occurred and that the failures will be larger than twenty-four. Thus with the observance of ten censored values of 24 and the knowledge that they are from an exponential distribution, an estimate of 55.802 is a reasonable estimate and may even be a low estimate.

Now, consider cases in which only one failure occurs. This occurrence happened in three of the data sets (data sets 20, 58 and 69 in Appendix A) generated. Table 5.5 summarizes the estimates with the data sets in which one failure is observed. In the first sample the failure occurs at time 11.580. Thus the data set consists of nine values censored at time 24, one value censored at time 12.420, and one value failed at time 11.580. The estimate of the mean survival time calculated using the Fuzzy-PLE is rather accurate ($\widetilde{S} = 49.328$) in this case relative to the PLE estimate of $P = 11.580$ and the pessimistic estimate of $S_{pes} = 21.818$. The 90% confidence interval of $\widetilde{S}$ covers 48, which is the actual value of the mean. In the second sample the failure occurred at time 10.111. Thus the data set consists of nine values censored at time 24, one value censored at time 13.889, and one value failed at time 10.111. The results are similar to the first case. In the third case the failure occurred at an earlier time of 0.441. Thus the data set here consists of nine values censored at time 24, one value censored at time 23.559, and one value failed at time 0.441. This smaller failure time results in an estimate very close to the estimates when the failure occurred much later (about half way through the observation period) and the estimate of vagueness is similar as well. The difference is in the length of the confidence intervals. The confidence interval for both $\tilde{S}$ and S are wider in this case. The estimate of mean survival time calculated using the Fuzzy-PLE is rather accurate in all three cases relative to the PLE estimates of 10.111, 11.580 and 0.441 respectively and the pessimistic estimate (S_{pes}) is 21.818 in all three cases.

Table 5.5 *exponential*(48) with One Failure

n	cen	$\tilde{S}$	$CI(\tilde{S})$	$se_B(\tilde{S})$	$\bar{S}$	S	$CI(S)$
11	10	49.328	(29.646, 55.649)	0.232	27.898	27.509	(13.321, 32.702)
11	10	49.338	(31.126, 55.920)	0.234	27.819	27.520	(11.390, 32.839)
11	10	49.402	(22.889, 57.708)	0.251	27.443	27.584	(9.597, 33.748)

5.3.1.2 Maximum Number of Failures. At the other extreme are the two data sets (data sets 11 and 65 in Appendix A) that resulted in eleven failures. The estimates for these cases are shown in Table 5.6. For these data sets the estimate of the mean survival time calculated using the PLE are 12.971 for the first data set, and 12.242 for the second data set.

Table 5.6 *exponential*(48) with Eleven Failures

n	cen	$\tilde{S}$	$CI(\tilde{S})$	$se_B(\tilde{S})$	$\bar{S}$	s	$CI(S)$
21	10	18.149	(12.387, 26.761)	0.505	7.069	6.720	(2.892, 12.746)
21	10	18.907	(12.868, 28.752)	0.537	7.920	7.479	(3.420, 14.369)

Table 5.7 *exponential*(48) Data with Eleven Failures

ti	di
0.309	0.0
0.492	1.0
1.638	1.0
4.402	0.0
4.912	0.0
5.602	0.0
6.463	0.0
7.556	0.0
9.826	0.0
10.166	1.0
11.833	1.0
12.167	0.0
13.834	0.0
14.174	1.0
16.444	1.0
17.537	1.0
17.960	0.0
18.286	0.0
18.398	1.0
24.0	1.0
24.0	1.0

Considering the first of these two cases as an example Table 5.7 shows the data set. Since the data is restricted to ten rows simulating ten units, observing eleven failures prior to time twenty-four is significant evidence supporting a mean failure time of less than twenty-four. With the largest failure at time 18.286 and more than half of the failures less than 8, a mean failure time greater than twenty-four is difficult to justify. The upper bound of the confidence interval is 26.761 and does not cover the population parameter of 48. Again this is an extreme case for the one hundred fifty data sets generated in this test and is not representative of the average case. Although $\tilde{S}$ does not accurately estimate the mean of the distribution it does out perform P in this case.

5.3.1.3 Analysis of the Estimates. Next an analysis of the estimates from the one hundred fifty data sets is considered. All one hundred fifty of these data sets are given in Appendix A. This summary provides insight into how the estimator performs on average in the circumstance of small and highly censored data sets. For each of the one hundred fifty data sets, Table 5.8 provides the number of values in the data set, and the estimates $\tilde{S}$, S_{pes}, and P respectively. Table 5.9 provides descriptive statistics of the one hundred fifty runs taken for the *exponential*(48) data sets.

The sizes of the data sets are important. By the nature of the simulation methodology, all of the data sets contain ten censored values. As discussed previously the extreme cases are the minimum case in which the data set contains ten censored values of 24 and the maximum cases is the observation of eleven failures resulting in the two data sets containing twenty-one values each. The average size for these simulations is 15 (mean = 15.08 and median = 15). Thus, on average the data sets contain about 67% censored values. For such small data sets this is a very high proportion of censored values.

Table 5.8 Estimates For Each of the *exponential*(48) Data Sets

Set	*n*	$\tilde{S}$	S_{pes}	*P*	*Set*	*n*	$\tilde{S}$	S_{pes}	*P*
1	15	31.183	16.000	17.196	39	17	27.013	14.118	10.897
2	13	39.465	18.462	10.426	40	14	34.348	17.143	18.978
3	14	33.883	17.143	21.326	41	17	25.282	14.118	14.068
4	17	24.870	14.118	16.039	42	15	29.566	16.000	19.638
5	17	27.737	14.118	12.844	43	14	33.134	17.143	20.847
6	15	30.216	16.000	17.992	44	14	33.042	17.143	20.763
7	15	31.054	16.000	17.567	45	19	21.748	12.632	14.133
8	12	42.401	20.000	20.112	46	16	27.075	15.000	17.652
9	17	26.956	14.118	11.445	47	14	36.769	17.143	7.504
10	17	26.073	14.118	12.383	48	13	37.637	18.462	21.203
11	21	18.149	11.429	12.972	49	14	36.038	17.143	11.917
12	16	26.269	15.000	18.681	50	16	28.797	15.000	13.296
13	15	31.268	16.000	14.831	51	17	26.797	14.118	13.601
14	17	24.896	14.118	14.354	52	16	29.783	15.000	12.768
15	13	37.516	18.462	16.389	53	15	29.961	16.000	19.137
16	15	31.080	16.000	14.397	54	14	33.036	17.143	18.627
17	12	44.302	20.000	6.329	55	20	20.074	12.000	14.487
18	12	42.546	20.000	13.724	56	19	21.161	12.632	15.739
19	16	29.164	15.000	12.022	57	15	31.114	16.000	15.271
20	11	49.328	21.818	11.580	58	11	49.338	21.818	10.111
21	17	26.439	14.118	14.429	59	16	28.269	15.000	16.203
22	17	25.295	14.118	14.050	60	17	25.155	14.118	16.388
23	15	31.266	16.000	18.258	61	12	42.622	20.000	21.445
24	15	30.242	16.000	20.218	62	15	29.161	16.000	16.370
25	17	25.935	14.118	13.606	63	14	35.248	17.143	16.658
26	17	25.340	14.118	13.529	64	15	30.604	16.000	18.009
27	16	27.020	15.000	17.589	65	21	18.907	11.429	12.215
28	13	37.302	18.462	21.473	66	15	29.915	16.000	18.227
29	18	24.843	13.333	12.138	67	16	26.342	15.000	19.321
30	16	27.136	15.000	18.766	68	16	27.563	15.000	15.398
31	15	30.578	16.000	18.889	69	11	49.402	21.818	0.441
32	14	32.989	17.143	20.761	70	15	29.349	16.000	19.376
33	18	24.355	13.333	15.106	71	20	21.359	12.000	12.698
34	15	30.287	16.000	16.417	72	12	43.550	20.000	19.944
35	12	44.316	20.000	3.323	73	13	40.136	18.462	9.483
36	14	33.930	17.143	19.605	74	18	25.723	13.333	11.315
37	16	28.577	15.000	16.581	75	17	26.067	14.118	15.624
38	12	42.616	20.000	22.326	76	14	36.671	17.143	10.365

Table 5.8 Continued

Set	*n*	$\tilde{S}$	S_{pes}	*P*	*Set*	*n*	$\tilde{S}$	S_{pes}	*P*
77	14	33.922	17.143	15.204	114	14	35.687	17.143	12.577
78	10	55.802	24.000	0.000	115	15	30.252	16.000	18.253
79	14	36.009	17.143	12.637	116	14	34.263	17.143	18.927
80	17	27.146	14.118	12.221	117	12	42.411	20.000	22.170
81	14	35.428	17.143	11.979	118	12	43.506	20.000	13.765
82	15	30.729	16.000	18.411	119	17	26.283	14.118	13.456
83	15	30.165	16.000	19.596	120	14	34.027	17.143	16.012
84	14	35.092	17.143	16.582	121	15	30.403	16.000	19.323
85	13	38.400	18.462	19.965	122	16	30.562	15.000	8.648
86	14	35.712	17.143	10.795	123	14	33.878	17.143	19.236
87	13	39.138	18.462	11.183	124	17	25.186	14.118	17.435
88	17	27.337	14.118	11.407	125	17	26.090	14.118	14.250
89	14	34.533	17.143	18.758	126	18	23.901	13.333	13.391
90	14	33.001	17.143	21.332	127	18	23.198	13.333	14.587
91	15	33.231	16.000	11.866	128	14	34.465	17.143	19.434
92	18	25.300	13.333	11.758	129	15	33.164	16.000	11.064
93	12	42.606	20.000	20.935	130	14	34.452	17.143	16.165
94	14	34.399	17.143	14.745	131	14	33.882	17.143	13.681
95	12	43.553	20.000	20.928	132	15	29.369	16.000	19.783
96	15	31.985	16.000	12.623	133	16	28.317	15.000	14.767
97	13	38.442	18.462	13.480	134	16	29.199	15.000	13.475
98	13	37.298	18.462	19.160	135	16	30.601	15.000	9.358
99	17	25.383	14.118	15.875	136	13	38.384	18.462	17.422
100	18	25.080	13.333	12.435	137	16	29.094	15.000	15.398
101	13	37.611	18.462	19.399	138	14	33.715	17.143	16.414
102	16	28.175	15.000	16.469	139	17	25.883	14.118	14.819
103	16	29.664	15.000	14.010	140	12	42.389	20.000	17.773
104	14	34.026	17.143	17.822	141	19	22.670	12.632	11.396
105	18	23.272	13.333	15.714	142	13	38.261	18.462	13.130
106	14	33.860	17.143	13.196	143	13	37.294	18.462	17.324
107	14	35.232	17.143	15.654	144	13	37.542	18.462	17.845
108	17	25.883	14.118	16.330	145	15	31.529	16.000	14.340
109	19	24.493	12.632	8.458	146	18	22.554	13.333	16.173
110	13	37.798	18.462	18.162	147	15	31.020	16.000	16.713
111	12	43.565	20.000	18.206	148	17	24.149	14.118	17.416
112	15	31.552	16.000	17.049	149	15	29.505	16.000	19.130
113	15	32.148	16.000	12.910	150	14	34.198	17.143	15.744

Table 5.9 *exponential*(48) Summary Statistics

Variable	*Mean*	*Median*	*StDev*	*Minimum*	*Maximum*
n	15.080	15.000	2.100	10.000	21.000
$\tilde{S}$	31.952	31.037	6.659	18.149	55.802
S_{pes}	16.226	16.000	2.288	11.429	24.000
P	15.386	15.726	3.937	0.000	22.326
$se_B(\tilde{S})$	0.672	0.690	0.135	0.000	0.916
l	20.938	20.605	4.964	12.387	55.802
u	43.616	43.663	6.213	26.761	57.708
$\bar{S}$	16.368	15.842	4.390	7.069	31.802
S	15.726	15.097	4.417	6.721	31.802
$se_B(S)$	0.527	0.545	0.114	0.000	0.762
lS	7.886	7.515	2.993	2.892	31.802
uS	24.624	24.661	4.297	12.746	34.146

Now the performance of P, the estimate of the mean survival time calculated using the Product-Limit Estimator, is considered. The minimum value of zero occurs with the data set that contains ten units censored at time 24. As stated previously this is actually a "no answer" result since no failures occur. The true minimum P estimate is 0.441. The maximum estimate of the mean survival time using the PLE is 22.326. The mean and the median of P are 15.386 and 15.726 respectively. Note that the pessimistic estimate (S_{pes}) has a mean and median larger than that of the estimates made using the PLE. The pessimistic estimate exceeded P in about half (seventy-three out of one hundred fifty) of the estimates. Figure 5.1 shows a histogram of the estimates made with the PLE. From this histogram it is seen that most of the estimates made using the PLE are between ten and twenty. Since 48 is being estimated, it could be concluded that the estimates made using the PLE do not perform well on these very small and highly censored data sets.

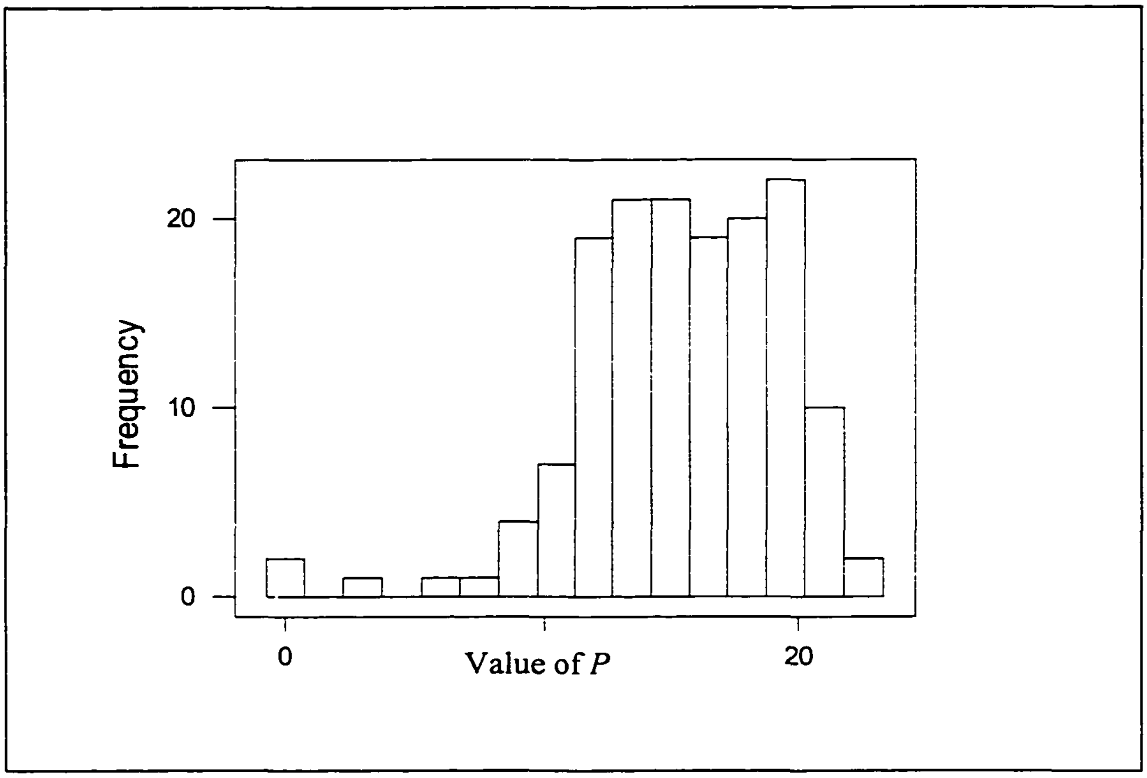


Figure 5.1 Histogram of P for the Data in Table 5.9

Since P does not seem to estimate well with these data sets, does $\tilde{S}$ do any better? The extreme cases have already been scrutinized, and it was observed that when no failures or only one failure occurred, the estimate was very accurate. It was also seen that on the other extreme that when eleven failures had occurred before time twenty-four, the estimates where not as accurate as seen in Table 5.6.

Now considering the average cases as shown in the second row of Table 5.9 it is seen that the mean of $\tilde{S}$ is 31.952 and the median of $\tilde{S}$ is 31.037 which are very good estimates considering they are based on observances made at time twenty-four. These are about twice the average estimate made using the PLE and closer to the population parameter of 48. It should also be noted that the standard deviation (column 4 of Table

5.9) is about 40% more for $\tilde{S}$ than for P. Thus on average $\tilde{S}$ out performs P and in fact it out performs the PLE in every sample, but the variability in the estimates are wider. The reason for this wider dispersion is that the PLE does not take into account properties of the censored times as does the Fuzzy-PLE. For example, the data sets {(1,1), (5, 1), (100, 0), (101, 1), (200, 1)} and {(90,1), (95, 1), (100, 0), (190, 1), (200, 1)} produce the same result (no variability) for the estimate made using the PLE (100 in both cases) but very different results for the estimate made using the Fuzzy-PLE (144.831 and 229.711 respectively). So overall the dispersion is larger for the Fuzzy-PLE but the dispersion is caused by the use of more information on the part of the Fuzzy-PLE. Figure 5.2 shows a histogram of the $\tilde{S}$ estimates for the *exponential*(48) data sets.

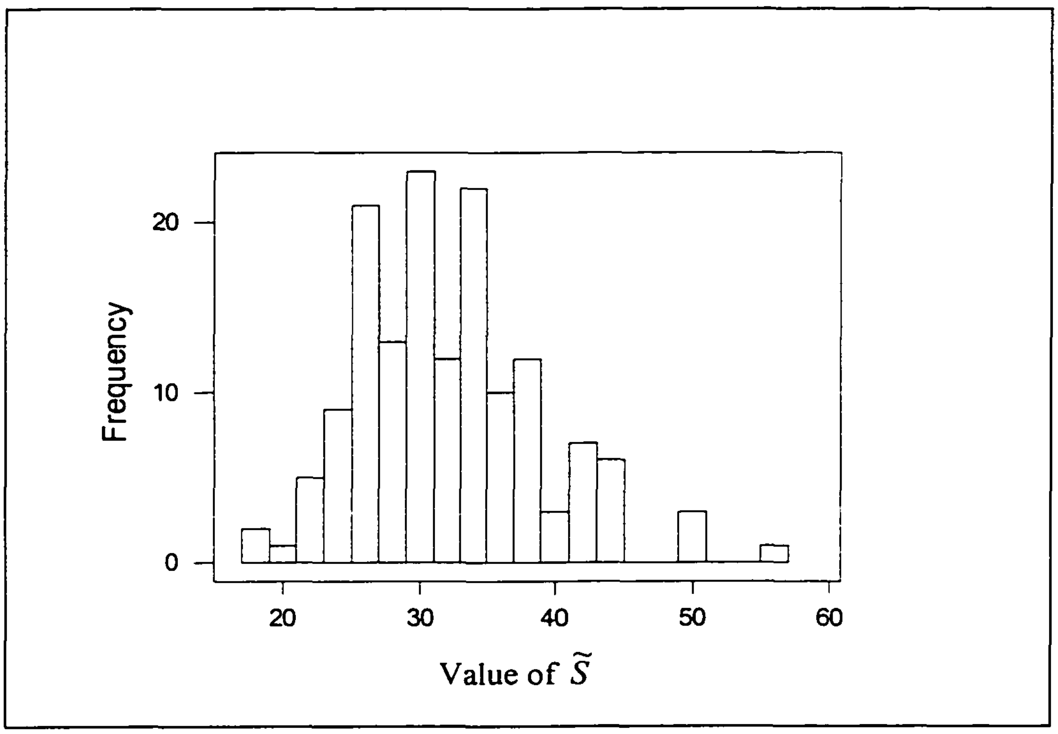


Figure 5.2 Histogram of $\tilde{S}$ for the Data in Table 5.9

Finally, the confidence intervals for the Fuzzy-PLE have a mean lower bound ($l\tilde{S}$) of 20.938 and mean upper bound ($u\tilde{S}$) of 43.616. The mean of the confidence intervals do not cover the mean of the distribution, but it needs to be recognized that the estimates are being made very early at time twenty-four. Of the one hundred fifty estimates, 26% of the confidence intervals actually covered the mean of 48. For completeness the summary statistics of the vagueness in the estimates are given in Table 5.6. Nothing unexpected is seen with these results. The average vagueness added to the average S_{pes} estimate is about the average $\tilde{S}$ estimate.

Next the *exponential*(24) distribution is considered. The data more matured at time twenty-four than for the *exponential*(48) data. The estimate of the mean survival time made using the PLE and the Fuzzy-PLE are closer in agreement than with the previous distribution, but there is still enough vague uncertainty to justify the use of the Fuzzy-PLE overt the PLE.

5.3.2 Simulations with the *exponential*(24) Distribution

For the *exponential*(24) distribution one-hundred data sets are randomly generated. Appendix B shows the one hundred data sets. In these simulations it is known that the population parameter being estimated is 24. Two of the data sets (data sets 71 and 92 in Appendix B) contain the minimum of fourteen values (four failure and ten censor times) and one (data set 31in Appendix B) contains the maximum of thirty-one values.

<u>5.3.2.1 Minimum Number of Failures.</u> Table 5.10 provides the estimates made with the two data sets containing fourteen values. The estimates S_{pes} and P for the first row are 17.143 and 18.125 respectively. For the second row they are 17.143 and 19.347. The estimate made using the PLE produces rather good results in both of these cases. The estimates made using the Fuzzy-PLE seems to produce an over estimate but in both cases the 90% confidence intervals cover the population parameter of 24. In addition, the amount of vagueness accounts for almost 50% of the estimate and provides a sense of the vague uncertainty toward the high end in the estimates.

Table 5.10 *exponential*(24) Data with Minimum Number of Failures

n	Cen	$\tilde{S}$	$CI(\tilde{S})$	$se_B(\tilde{S})$	$\bar{S}$	S	$CI(S)$
14	10	33.877	(23.080, 46.555)	0.707	17.218	16.734	(9.025, 26.466)
14	10	33.013	(23.128, 45.429)	0.688	16.440	15.870	(8.610, 25.997)

<u>5.3.2.2 Maximum Number of Failures.</u> At the other extreme is the one case containing thirty-one values. Again, as in the *exponential*(48), with so many failures an underestimate of the mean is made. In this case P estimates the mean survival time as 9.4748 and the pessimistic estimate is 7.7419. Table 5.11 gives the estimates made for this data set.

Table 5.11 *exponential*(24) Data with Maximum Number of Failures

n	Cen	$\tilde{S}$	$CI(\tilde{S})$	$se_B(\tilde{S})$	$\bar{S}$	S	$CI(S)$
31	10	10.816	(7.725, 16.517)	0.338	3.320	3.075	(1.355, 6.793)

Table 5.12 provides the data set. With sixteen of the twenty-one failures less than 10, twenty-seven of the thirty-one values less than 14.6, and the maximum censored value of 24, there is much evidence in this case that the mean survival time is in the neighborhood of the 10.816 estimated using the Fuzzy-PLE.

Table 5.12 *exponential*(24) Data Set Containing Thirty-one Values

t_i	d_i
0.013	0.0
0.354	1.0
0.462	0.0
0.505	0.0
0.561	0.0
0.822	0.0
0.993	0.0
1.489	0.0
2.737	0.0
3.108	1.0
3.369	0.0
3.570	0.0
4.734	1.0
5.933	0.0
5.974	0.0
6.176	1.0
6.515	0.0
7.927	1.0
7.975	0.0
8.049	0.0
9.419	0.0
10.259	1.0
10.853	1.0
11.594	0.0
11.891	0.0
13.741	0.0
14.581	1.0
18.788	0.0
20.070	0.0
23.538	1.0
24.0	1.0

5.3.2.3 Analysis of the Estimates. Table 5.13 provides the descriptive statistics of the estimates made with the one-hundred *exponential*(24) data sets and Table 5.14 shows the number of values in each of the data sets, and the estimates $\tilde{S}$, S_{pes}, and P respectively. The median number of data points is 19, just four more than with the *exponential*(48) data sets, but the estimate of the mean survival time made using the Fuzzy-PLE is much closer to the population parameter of 24. The confidence intervals cover 24 on average. In fact, only eighteen of the one hundred do not cover 24. These estimates have proportionally less vague uncertainty on average compared to the estimates made with the *exponential*(48) data. The ratio of the means of S to $\tilde{S}$ is 0.4864 for the *exponential*(48) data and 0.4076 for the *exponential*(24) data.

Table 5.13 *exponential*(24) Summary Statistics

Variable	*Mean*	*Median*	*StDev*	*Minimum*	*Maximum*
n	19.860	19.000	3.185	14.000	31.000
$\tilde{S}$	21.216	21.430	4.971	10.816	33.877
S_{pes}	12.383	12.632	1.921	7.742	17.143
P	14.030	14.126	2.573	6.220	19.550
$se_B(\tilde{S})$	0.602	0.615	0.150	0.208	0.961
l	14.267	14.228	3.139	7.725	23.128
u	31.116	31.513	6.890	16.422	46.554
$\bar{S}$	9.341	9.322	3.250	3.320	17.255
S	8.833	8.735	3.139	3.074	16.734
$se_B(S)$	0.459	0.474	0.134	0.161	0.763
lS	4.205	4.047	1.609	1.355	9.025
uS	16.159	16.087	4.814	5.612	26.726

Table 5.14 Estimates For Each of the *exponential*(24) Data Sets

Set	*n*	$\tilde{S}$	S_{pes}	*P*	*Set*	*n*	$\tilde{S}$	S_{pes}	*P*
1	19	20.961	12.632	14.621	39	21	18.471	11.429	12.023
2	24	17.457	10.000	9.323	40	24	17.007	10.000	10.674
3	18	23.886	13.333	15.345	41	20	22.835	12.000	10.285
4	27	12.337	8.889	11.815	42	24	14.794	10.000	12.958
5	22	17.936	10.909	13.124	43	24	15.315	10.000	13.006
6	16	28.949	15.000	16.003	44	18	23.514	13.333	15.820
7	22	16.381	10.909	14.224	45	20	19.778	12.000	13.474
8	17	24.002	14.118	16.280	46	24	15.447	10.000	12.973
9	17	25.065	14.118	16.990	47	23	15.648	10.435	13.187
10	22	17.016	10.909	13.607	48	18	25.013	13.333	10.901
11	19	22.974	12.632	12.274	49	16	27.259	15.000	18.459
12	16	26.982	15.000	18.306	50	21	18.063	11.429	12.544
13	17	25.201	14.118	17.727	51	18	26.001	13.333	10.279
14	24	13.273	10.000	13.609	52	18	22.376	13.333	14.673
15	19	22.611	12.632	12.763	53	18	23.919	13.333	14.281
16	21	17.867	11.429	14.735	54	21	21.683	11.429	9.980
17	22	17.071	10.909	14.041	55	21	16.293	11.429	14.957
18	16	26.993	15.000	17.975	56	25	14.346	9.600	12.622
19	20	19.871	12.000	14.011	57	15	30.516	16.000	16.592
20	15	32.156	16.000	15.256	58	19	23.360	12.632	12.426
21	21	18.808	11.429	14.299	59	20	18.970	12.000	16.787
22	24	13.912	10.000	13.550	60	18	23.619	13.333	15.564
23	17	24.208	14.118	18.419	61	19	22.461	12.632	14.611
24	19	20.162	12.632	15.544	62	17	25.086	14.118	16.651
25	17	26.110	14.118	15.217	63	18	23.886	13.333	14.363
26	22	16.526	10.909	14.715	64	24	17.286	10.000	10.536
27	19	21.875	12.632	14.698	65	16	27.138	15.000	13.959
28	24	15.181	10.000	11.623	66	17	25.951	14.118	17.007
29	19	20.953	12.632	14.519	67	18	21.836	13.333	16.142
30	23	15.314	10.435	12.148	68	22	18.956	10.909	10.441
31	31	10.816	7.742	9.475	69	17	25.344	14.118	14.239
32	17	24.321	14.118	18.600	70	18	23.437	13.333	14.265
33	15	30.584	16.000	16.229	71	14	33.877	17.143	18.125
34	18	22.650	13.333	16.309	72	23	16.296	10.435	12.043
35	18	22.529	13.333	16.756	73	21	19.678	11.429	10.320
36	18	23.560	13.333	14.550	74	20	20.560	12.000	13.479
37	19	21.177	12.632	14.025	75	27	13.911	8.889	10.063
38	19	23.325	12.632	12.568	76	16	31.457	15.000	6.220

Table 5.14 Continued

Set	*n*	$\tilde{S}$	S_{pes}	*P*	*Set*	*n*	$\tilde{S}$	S_{pes}	*P*
77	24	15.158	10.000	11.510	89	21	17.760	11.429	12.798
78	20	20.006	12.000	14.210	90	19	23.949	12.632	9.847
79	24	13.838	10.000	13.793	91	18	22.515	13.333	16.989
80	19	23.326	12.632	12.977	92	14	33.013	17.143	19.347
81	21	17.964	11.429	13.336	93	27	13.171	8.889	11.702
82	21	18.754	11.429	14.658	94	21	16.535	11.429	15.669
83	19	22.478	12.632	12.016	95	21	18.063	11.429	13.714
84	16	28.932	15.000	13.498	96	24	14.585	10.000	13.770
85	17	24.823	14.118	17.539	97	21	21.121	11.429	7.624
86	22	18.612	10.909	12.740	98	18	22.244	13.333	16.834
87	21	17.853	11.429	15.508	99	15	30.266	16.000	19.550
88	19	20.976	12.632	14.664	100	18	23.307	13.333	16.481

A comparison of the confidence intervals for the two distributions shows that the *exponential*(24) data produced smaller intervals on average. The length of the mean length of the confidence intervals form the *exponential*(48) data is 22.678, and the mean length of the confidence intervals from the *exponential*(24) data is 16.849. Thus in the comparison of the results of the estimates made from the *exponential*(48) and *exponential*(24) data it is seen what may be expected. That is, with more mature data the estimates are more accurate, less variable, with shorter confidence intervals, and contain less vague uncertainty.

Next, the *exponential*(6) distribution is considered. The data are well matured at time twenty-four, and the results are rather consistent across samples. The estimate of the mean survival time made using the PLE and the Fuzzy-PLE are closer in agreement than with the previous distributions.

<u>5.3.3 Simulations with the *exponential*(6) Distribution</u>

For the *exponential*(6) distribution the seventy-five data sets shown in Appendix C are randomly generated. In these simulations it is known that the population parameter being estimated is 6. Of the seventy-five randomly generated data sets one (data set 74 in Appendix C) had the maximum of sixty-eight points and one (data set 49 in Appendix C) had the minimum of thirty-two data points. Again, as with the *exponential*(48) and *exponential*(24) data sets, the largest data set produced the smallest underestimate and the smallest data set produced the largest overestimate. The results for these two cases are shown in Table 5.15

Table 5.15 The Extreme Cases for the *exponential*(6) Data

n	cen	$\tilde{S}$	$CI(\tilde{S})$	$se_B(\tilde{S})$	$\bar{\tilde{S}}$	S	$CI(S)$
68	10	3.894	(3.204, 4.668)	0.0392	0.377	0.365	(0.197, 0.629)
32	10	11.947	(7.720, 20.310)	0.459	4.764	4.447	(1.913, 10.138)

The mean survival time calculated using the PLE for the case with 68 data points is 4.099 and for the case with 32 data points is 8.059. Thus in both cases the PLE has out performed the Fuzzy-PLE. The more interesting aspect here is the confidence interval in the case with 32 data points. This case does not cover the mean of 6, but it is an overestimate, not an underestimate, as with the previous two distributions. This observation may be an indicator that with this much data (median number of data points for the exponential(6) data are 50 and 20% censoring) the PLE estimate should also be taken into consideration. Looking at the summary statistics of these simulations reveal that on average the Fuzzy-PLE still outperforms the PLE when estimating the mean survival time in these cases.

<u>5.3.3.1 Analysis of the Estimates.</u> Table 5.16 shows the summary statistics for the simulations on the *exponential*(6) data sets. The mean of $\tilde{S}$ is 6.071 while the mean of the estimates made using the PLE is 5.755. Both are very close to the mean of the distribution of 6.

Table 5.16 *exponential*(6) Summary Statistics

Variable	*Mean*	*Median*	*StDev*	*Minimum*	*Maximum*
n	50.120	50.000	7.130	32.000	68.000
$\tilde{S}$	6.071	5.840	1.357	3.894	11.947
S_{pes}	4.892	4.800	0.757	3.529	7.500
P	5.755	5.546	0.959	4.099	8.343
$se_B(\tilde{S})$	0.122	0.114	0.071	0.038	0.459
l	4.670	4.543	0.825	3.204	7.720
u	8.429	8.192	2.556	4.668	20.310
$\bar{S}$	1.289	1.144	0.764	0.377	4.764
S	1.179	1.064	0.686	0.365	4.447
$se_B(S)$	0.078	0.069	0.059	0.000	0.337
lS	0.510	0.467	0.264	0.197	1.913
uS	2.677	2.568	1.601	0.629	10.138

As with the other two distributions the variance of $\tilde{S}$ is larger than the variance of P. In these cases the minimum of the $\tilde{S}$ estimates is less than the minimum for the P estimates and the maximum of the $\tilde{S}$ is greater than the maximum for the P estimates. In the *exponential*(48) and *exponential*(24) the minimum P estimate is less than the minimum $\tilde{S}$ estimate. In other words, in the previous distributions $\tilde{S}$ consistently had larger estimates than P. In fact there was only one data set from the *exponential*(24) distribution in which P resulted in a larger estimate than $\tilde{S}$ and no cases with the *exponential*(48) distribution. Of the seventy-five *exponential*(6) data sets twenty-nine of them resulted in P having a larger estimate than the $\tilde{S}$ and another thirty in which the $\tilde{S}$ exceeded P by 0.3 or less as seen in Table 5.17. In addition, the ratio of S to $\tilde{S}$ is 0.1942 and recall that for the *exponential*(24) this ratio is 0.4076 and for the *exponential*(48) this ratio is 0.4864. Thus, with *exponential*(6) data sets significantly less vague uncertainty is quantified in the estimates. Again this may be an indication that there exists sufficient data in these cases to use the PLE instead of the Fuzzy-PLE when making the estimate of the mean survival time.

Finally, looking at the confidence intervals it is seen that on average the actual mean is covered. Actually, sixteen of the seventy-five did not cover the mean of the distribution. The point of interest is that the lower bound on six of the confidence intervals is greater than the mean of the distribution, indicating an overestimate. As stated previously, this situation only occurred once with the *exponential*(24) data and not at all with the *exponential*(48) data. The mean length of the confidence intervals for the *exponential*(6) distribution data sets is 3.759 as compared with 16.849 for the *exponential*(24) data sets and 22.678 for the *exponential*(48) data sets. The shorter confidence intervals may be yet another indication that sufficient data may exist to consider using standard statistical methods.

Table 5.17 Estimates For Each of the *exponential*(6) Data Sets

Set	*n*	$\tilde{S}$	S_{pes}	*P*	*Set*	*n*	$\tilde{S}$	S_{pes}	*P*
1	53	5.145	4.528	5.389	39	59	4.609	4.068	4.775
2	58	4.641	4.138	4.871	40	44	6.874	5.455	6.714
3	48	6.531	5.000	5.582	41	48	5.795	5.000	6.069
4	54	5.146	4.444	4.977	42	39	8.881	6.154	6.721
5	34	9.540	7.059	8.343	43	48	5.840	5.000	5.931
6	56	4.802	4.286	5.110	44	50	5.525	4.800	5.963
7	49	5.975	4.898	5.546	45	59	4.522	4.068	4.770
8	48	6.304	5.000	5.705	46	52	5.756	4.615	5.208
9	45	6.236	5.333	6.452	47	59	4.867	4.068	4.831
10	46	7.277	5.217	5.571	48	55	4.913	4.364	5.242
11	62	4.588	3.871	4.307	49	32	11.947	7.500	8.059
12	45	6.928	5.333	6.153	50	47	5.838	5.106	6.310
13	48	6.643	5.000	5.758	51	59	4.532	4.068	4.743
14	48	5.812	5.000	6.048	52	38	7.989	6.316	7.805
15	46	7.956	5.217	5.103	53	48	6.359	5.000	5.754
16	51	5.822	4.706	5.611	54	54	5.694	4.444	4.841
17	52	5.680	4.615	5.255	55	38	8.545	6.316	7.529
18	64	4.283	3.750	4.435	56	50	5.827	4.800	5.579
19	43	6.922	5.581	6.855	57	47	6.153	5.106	6.105
20	55	5.055	4.364	5.428	58	44	6.710	5.455	6.929
21	56	4.845	4.286	4.908	59	57	4.742	4.211	5.015
22	45	6.631	5.333	6.056	60	50	6.592	4.800	5.308
23	56	5.072	4.286	5.063	61	48	6.183	5.000	5.640
24	63	4.408	3.810	4.325	62	62	4.535	3.871	4.495
25	55	6.187	4.364	4.879	63	44	6.760	5.455	6.915
26	52	5.403	4.615	5.359	64	53	5.890	4.528	5.169
27	53	6.110	4.528	5.368	65	50	5.767	4.800	5.768
28	55	5.081	4.364	5.208	66	58	4.996	4.138	4.671
29	50	5.727	4.800	5.418	67	41	7.287	5.854	7.077
30	56	5.453	4.286	5.365	68	53	5.456	4.528	5.343
31	41	7.276	5.854	7.207	69	46	6.103	5.217	6.392
32	49	6.147	4.898	6.022	70	48	6.029	5.000	6.326
33	37	9.865	6.486	7.668	71	51	5.885	4.706	5.462
34	54	5.404	4.444	5.122	72	47	6.293	5.106	6.539
35	40	7.113	6.000	7.637	73	46	6.404	5.217	6.189
36	54	5.007	4.444	5.365	74	68	3.894	3.529	4.099
37	54	5.207	4.444	5.365	75	53	5.846	4.528	4.840
38	39	7.290	6.154	7.644					

5.3.4 Conclusions

In this chapter simulations are performed to test the performance of the Fuzzy PLE. Data is randomly generated from several distributions and censored and failure times are extracted from this data. Estimates are made from the resulting data set using the Fuzzy-PLE and associated uncertainty measures. Multiple runs are made for each distribution, the data is analyzed, and comparisons are made with the actual mean of the distribution and the performance of the PLE on each data set. It is demonstrated that as more data is made available that the Fuzzy-PLE produced better estimates on average with smaller confidence intervals. It is also seen that when little data is available that the Fuzzy-PLE out performs the PLE in making the estimates.

The data are generated from the *exponential*(48), *exponential*(24), and *exponential*(6) distributions. All of the estimates are made at time twenty-four. Thus, the estimate made at time twenty-four with the *exponential*(48) distributed data is made very early, the data have not matured and therefore contain a high percentage of censored values. The estimate made at time twenty-four with the *exponential*(6) distributed data is made with very mature data, and it is seen that the estimates of the mean survival time made using the PLE are about as good as those made using the Fuzzy-PLE. The estimates made using the *exponential*(24) distributed data are given to show the results with a moderate amount of censoring in the data set. In this case, on average, about 52% of the data are censored, and it is seen that the estimates of the mean survival time made using the Fuzzy-PLE are very accurate and closer to the mean of the distribution than the estimates made using the PLE.

In this chapter data sets were generated from different distributions and estimates were all made at the same time of twenty-four. In the next chapter, a data set is generated from an *exponential*(24) distribution and estimates are made at different times for one data set. The goal is to simulate the tracking of in-service equipment over time and to further investigate at which point sufficient data may exist to use the PLE instead of the Fuzzy-PLE in making the estimates.

CHAPTER 6

SIMULATIONS OVER TIME

6.1 Introduction

The simulation of this chapter is developed to model the situation that has motivated this research. This simulation consists of simulating the failure times of several pieces of equipment over time. Units are put into service, failures are recorded, and at given times during the simulation estimates are made of the mean survival time. Several issues are addressed with these simulations. It is demonstrated that for early times the estimate of the mean survival time obtained using the Fuzzy-PLE is superior to the estimate obtained using the PLE and that with matured data the estimate obtained using the Fuzzy-PLE agrees with that of the PLE.

Initially, all of the estimates are made with the parameters *ek* and *uc* set to zero. These values indicate that no expert knowledge is considered in making the estimates. Also the parameter *U* is set to 0.65. This is a moderate value of *U* used in the development of the Fuzzy-PLE in chapter three. Varying these parameters is then considered separately. The same data used in the simulation of section 5.3 is considered with different values of *ek*, *uc*, and *U*. Comparisons are then made, and an understanding of the proper use of these parameters is established.

6.2 Simulating Over Time

The simulations of Chapter 5 generated data sets from several distributions and made estimates at time twenty-four in all cases. In this chapter it is desired to test the estimator from another perspective. Here, a data set is generated representing ten units and then samples are taken from the data set at different times. The simulation models the use of in-service equipment and demonstrates how the accuracy of the estimates changes over time. It is seen that the estimate of the mean survival time calculated using the Fuzzy-PLE more accurately estimates the actual mean early and coincides with the estimate made using the PLE at the later times.

To test the estimators and run simulations for the scenario described above, data are randomly generated from an *exponential*(24) distribution. Since the estimator is non-parametric the underlying distribution is not important. Reliability engineers suggest the use of the exponential distribution. The justification for the use of this distribution is that the testing during development of the equipment eliminated the left side of the bathtub curve and maintenance eliminates the right side, resulting in a constant hazard rate. Thus, the exponential distribution is justified.

In this chapter, ten rows of data are randomly generated from an *exponential*(24) distribution as in Chapter 5 except that now the data are generated such that the row sums are greater than or equal to 100. Therefore, at any given time between zero and one hundred, the failure and censor times can be calculated, survival curves estimated, and estimates of the mean survival time with associated uncertainty estimates can be made. In the example of this section, starting at time four and at intervals of four across the time axis, a sample is extracted that represents the information known at the given time. From each of the samples the estimates are made.

Table 6.1 shows the data set used in this simulation. The column of zeros represents the starting time of each of the ten units. As stated previously the data are randomly generated from an *exponential*(24) distribution. The actual mean and the standard deviation of this sample (excluding the column of zeros) are 23.915 and 22.494 respectively.

The first estimate is made at time four. The data set contains the thirteen censored and failure times listed in Table 6.2. Table 6.3 gives the data set at time eight. No additional failures have occurred and the censored times have simply increased by four.

Table 6.1 Data for the Simulation Over Time

0	38.450	5.711	49.973	14.508	6.237	4.934	2.788	21.573	31.786
0	49.890	4.810	23.535	15.574	14.062	2.586	85.130	16.704	10.464
0	2.691	47.920	2.121	0.016	3.844	31.481	67.822	3.211	31.996
0	36.110	93.792	82.467	3.413	14.836	35.335	32.615	5.916	18.633
0	13.910	6.959	19.225	46.068	3.360	3.209	51.674	12.728	25.087
0	20.610	2.472	35.417	30.180	35.508	4.048	16.881	30.610	17.788
0	3.505	6.339	44.495	0.638	23.794	76.720	72.242	3.221	21.759
0	31.660	11.053	16.555	7.998	80.464	40.659	26.058	28.783	86.135
0	9.197	9.170	6.761	10.522	22.120	19.204	15.901	3.437	12.386
0	1.009	16.614	37.062	7.359	26.978	2.677	21.265	39.767	20.133

Table 6.2 Data Set at Time = 4

t_i	d_i
0.4950	1.0
1.0090	0.0
1.3090	1.0
2.6910	0.0
2.9910	1.0
3.5050	0.0
4.0000	1.0
4.0000	1.0
4.0000	1.0
4.0000	1.0
4.0000	1.0
4.0000	1.0
4.0000	1.0

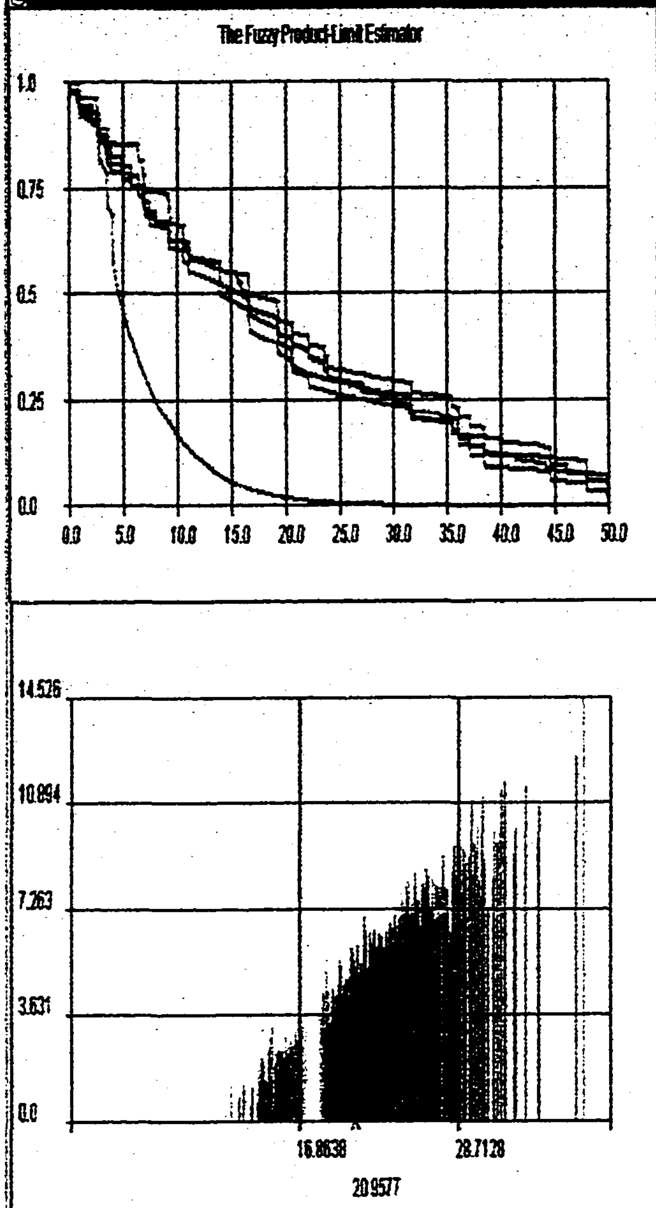
Table 6.3 Data Set at Time = 8

t_i	d_i
1.0090	0.0
2.6910	0.0
3.5050	0.0
4.4950	1.0
5.3090	1.0
6.9910	1.0
8.0000	1.0
8.0000	1.0
8.0000	1.0
8.0000	1.0
8.0000	1.0
8.0000	1.0
8.0000	1.0

Figure 6.1 shows a screenshot from the computer program that performed this simulation. On the upper left are the graphs of the survival curves estimated using the Fuzzy-PLE. The graphs are color coded to the times listed on the table at the right hand

side of the figure. For this example the graphs are given only for the estimates made at times 4, 24, 44, 64, 84, and 100. The bottom left shows the confidence interval for the last estimate given at time 100.

The graph for the survival curve at time 4 is red. The "probability" of survival remains relatively high (above 60%) up to time 4 and then drops off rapidly after time 4. At time 5 the probability of survival is estimated at about 40% and at time 10 the probability is down to about 13%. The estimate of survival at time 24 is virtually zero based on the information available at time 4. As seen in the table to the right side of Figure 6.1, the estimate of the mean survival time using the PLE is 3.222. The estimate of the mean survival time using the Fuzzy-PLE is 6.263, nearly twice that of the PLE estimate.



Advance

TIME	n	P	S~	S	seS(S~)	C(S~)	C(S)
*4	13	3.222	6.263	3.186	0.1059	(4.2182, 8.2313)	(1.5755, 4.7689)
*8	13	3.25	13.366	7.212	0.1974	(8.0221, 16.8954)	(3.1545, 8.9656)
[illegible]	15	7.514	16.01	8.01	0.4078	(10.0196, 22.6877)	(4.0235, 13.2891)
[illegible]	16	10.72	18.852	8.851	0.6261	(12.0747, 27.246)	(4.236, 15.2271)
	18	12.123	20.165	9.053	0.5154	(13.2675, 30.0792)	(4.3147, 16.4484)
	21	13.16	19.416	7.987	0.6658	(12.7687, 29.8329)	(3.785, 15.6156)
	22	13.211	21.707	8.979	0.5801	(14.0255, 33.6645)	(3.9503, 17.5948)
[illegible]	23	17.81	22.095	8.181	0.7665	(14.5592, 34.5924)	(3.74, 16.9793)
[illegible]	24	17.674	24.447	9.447	0.7646	(15.2805, 37.8558)	(3.6795, 18.9757)
*40	26	20.433	21.583	6.198	0.7513	(15.0682, 30.7484)	(2.6624, 12.1709)
*44	28	19.677	22.444	6.729	0.617	(15.7919, 34.2422)	(3.1443, 14.8914)
*48	29	19.763	23.773	7.221	0.7921	(16.4459, 35.8597)	(2.9474, 16.2145)
*52	31	22.563	21.102	4.327	0.4	(15.8748, 27.7682)	(2.2022, 8.2548)
*56	37	20.347	18.153	3.017	0.2417	(14.3872, 23.6342)	(1.6916, 5.356)
*60	41	19.103	17.17	2.535	0.1962	(13.5891, 21.7796)	(1.3954, 4.1041)
*64	42	19.564	17.808	2.569	0.2613	(13.9568, 22.0392)	(1.3858, 4.3547)
*68	43	19.97	18.442	2.628	0.2352	(14.5655, 22.6265)	(1.4434, 4.6551)
*72	43	20.689	19.573	2.828	0.186	(15.4847, 23.9756)	(1.4861, 5.0581)
*76	43	21.468	20.879	3.204	0.2072	(16.2346, 25.8647)	(1.5087, 5.7784)
*80	46	20.634	20.488	3.096	0.3446	(16.2193, 25.1418)	(1.5171, 6.0409)
*84	46	21.382	21.837	3.576	0.4637	(16.8795, 27.6151)	(1.6502, 7.5081)
*88	47	21.864	23.045	4.321	0.3185	(18.1614, 31.0277)	(2.0461, 10.4201)
*92	52	20.529	21.674	3.981	0.4407	(17.2281, 29.4231)	(1.7653, 9.9807)
*96	56	19.939	20.496	3.353	0.339	(16.336, 28.0782)	(1.4832, 9.2648)
*100	57	20.232	20.957	3.413	0.3701	(16.8638, 28.7128)	(1.5615, 9.4382)

Figure 6.1 Screenshot of Simulation Over Time

The graphs of the survival curves estimated at times 4, 24, and 44 are shown in Figure 6.2. Notice the scale is different from the scale on the graph in Figure 6.1. The scale here is from zero to one hundred. The survival curves estimated at time 24 and 44 coincide after time 40, slowly sloping from about 0.12 to 0.

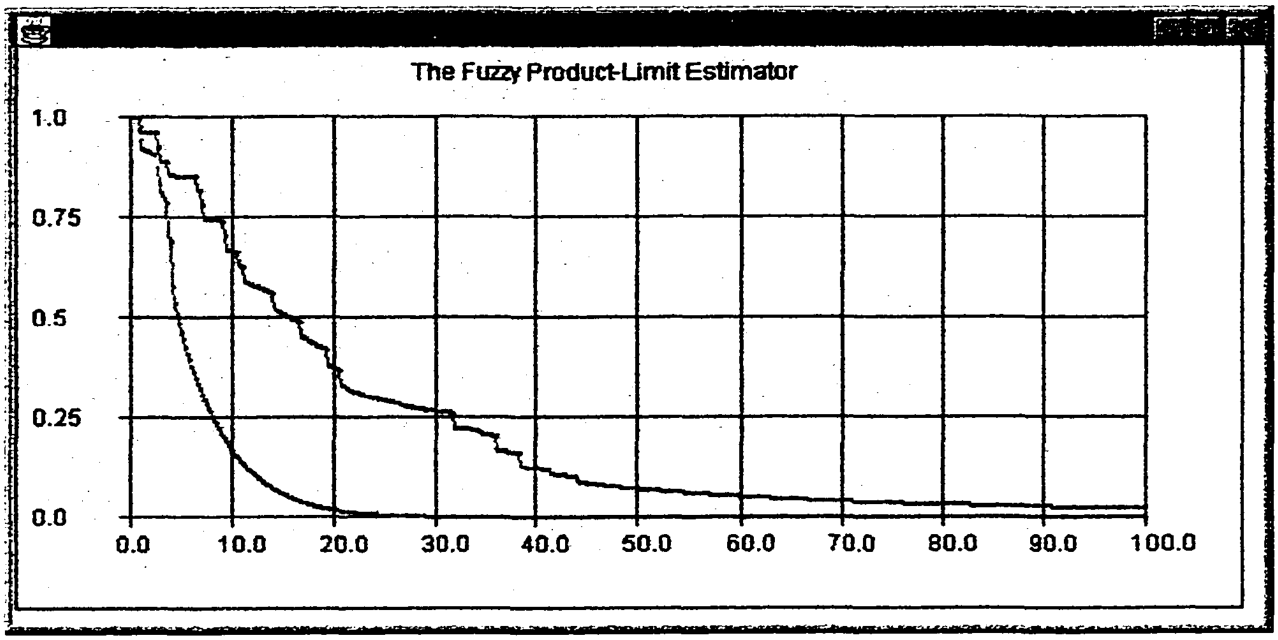


Figure 6.2 Survival Curves at Times 4, 24, and 44

Figure 6.3 shows the survival curves for the estimates made at times 44, 64 and 100. It is seen that the curves estimated at times 64 and 100 do not coincide with the curve estimated at time 44. These two curves drop off sharply after time forty. The curve estimated at time 64 is essentially zero after time fifty and the curve estimated at time 100 is essentially zero after time seventy. This set of graphs shows how, over time, the estimates of the survival curves evolve as more data becomes available.

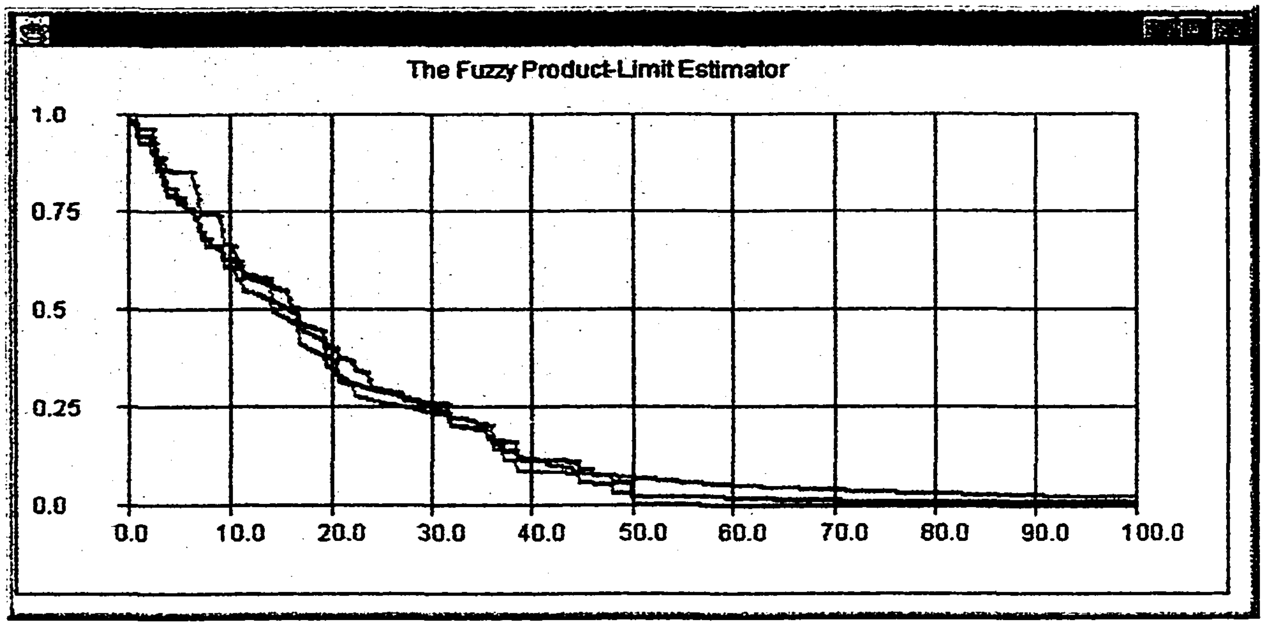


Figure 6.3 Survival Curves for Times 44, 64, and 100

Now the estimates of the mean survival time and associated uncertainty measures are considered. The information from the table to the left side of Figure 6.1 is repeated in Table 6.4 for convenience. The number of data points n used to make the estimates range from 13 (77% censored) to 57 (17.5% censored). The Fuzzy-PLE makes a good estimate as early as time 16 with just 16 data points (62.5% censored). The Fuzzy-PLE estimates the mean survival time as 18.852 and has a confidence interval of (12.0747, 27.2460). Recall that the actual mean and standard deviation of this data set is 23.915 and 22.494 respectively. Thus the actual mean and standard deviation are covered by the confidence interval.

Table 6.4 Results of Simulation in Figure 6.1

Time	*n*	*P*	$\tilde{S}$	*S*	$seB(\tilde{S})$	CI($\tilde{S}$)	CI(*S*)
4	13	3.222	6.263	3.186	0.1059	(4.2182, 8.2313)	(1.5755, 4.7689)
8	13	3.250	13.366	7.212	0.1974	(8.0221, 16.8954)	(3.1545, 9.9656)
12	15	7.514	16.010	8.010	0.4078	(10.0196, 22.6877)	(4.0235, 13.2891)
16	16	10.720	18.852	8.851	0.6261	(12.0747, 27.2460)	(4.2360, 15.2271)
20	18	12.123	20.165	9.053	0.5154	(13.2675, 30.0792)	(4.3147, 16.4484)
24	21	13.160	19.416	7.987	0.6658	(12.7687, 29.8329)	(3.7850, 15.6156)
28	22	13.211	21.707	8.979	0.5801	(14.0255, 33.6645)	(3.9503, 17.5948)
32	23	17.810	22.095	8.181	0.7665	(14.5592, 34.5924)	(3.7400, 16.9793)
36	24	17.674	24.447	9.447	0.7646	(15.2005, 37.8558)	(3.6795, 18.9757)
40	26	20.433	21.583	6.198	0.7513	(15.0682, 30.7484)	(2.6624, 12.1709)
44	28	19.677	22.444	6.729	0.6170	(15.7919, 34.2422)	(3.1443, 14.8914)
48	29	19.763	23.773	7.221	0.7921	(16.4459, 35.8597)	(2.9474, 16.2145)
52	31	22.563	21.102	4.327	0.4000	(15.8748, 27.7602)	(2.2022, 8.2548)
56	37	20.347	18.153	3.017	0.2417	(14.3872, 23.6342)	(1.6916, 5.3560)
60	41	19.103	17.170	2.535	0.1962	(13.5091, 21.7796)	(1.3954, 4.1041)
64	42	19.564	17.808	2.569	0.2613	(13.9569, 22.0392)	(1.3858, 4.3547)
68	43	19.970	18.442	2.628	0.2352	(14.5655, 22.6265)	(1.4434, 4.6551)
72	43	20.609	19.573	2.828	0.1960	(15.4847, 23.9756)	(1.4861, 5.0581)
76	43	21.468	20.879	3.204	0.2072	(16.2346, 25.8647)	(1.5087, 5.7784)
80	46	20.634	20.488	3.096	0.3446	(16.2193, 25.1418)	(1.5171, 6.0409)
84	46	21.382	21.837	3.576	0.4637	(16.8795, 27.6151)	(1.6502, 7.5001)
88	47	21.864	23.045	4.321	0.3185	(18.1614, 31.0277)	(2.0461, 10.4201)
92	52	20.529	21.674	3.981	0.4407	(17.2281, 29.4231)	(1.7653, 9.9807)
96	56	19.939	20.496	3.353	0.3390	(16.3360, 28.0782)	(1.4832, 9.2648)
100	57	20.232	20.957	3.413	0.3701	(16.8638, 28.7128)	(1.5615, 9.4382)

The PLE makes its first good estimate at time 40 with just 26 data points (38.5% censored). The estimate of 20.433 has an associated 90% confidence interval (obtained using normal probability methods) of (15.2134, 25.6544), which covers the true mean of the distribution. Before time 40 the PLE underestimates as expected and the associated confidence intervals did not cover the mean of the distribution. After time 40 the estimates made using the PLE are relatively consistent with only a difference of 3.46

between the highest estimate of 22.563 occurring at time 52 and lowest estimate of 19.103 occurring at time 60.

The mean survival time estimated using the Fuzzy-PLE quickly approaches the actual mean of the data with an estimate of 16.010 at time 12 using only 15 data points (67% censored) and a confidence interval of (10.0196, 22.6877). The Fuzzy-PLE reaches a high estimate of 24.447 at time 36. The next estimate, at time 40, is lower and looking down the column several rises and dips occur in the estimates.

Comparing the estimates of the mean survival time using the Fuzzy-PLE after time 40 as was done with the estimates made using the PLE it is seen that the difference between the largest estimate of 23.773 and the smallest estimate of 17.170 is 6.603. This amount is larger than that observed with the PLE estimates but still relatively small. It is noted that the smallest with both estimates occurs at time 60. After time 60 the largest difference at any given time between the estimate made using the PLE and that made using the Fuzzy-PLE is 1.810. Figure 6.4 is a plot of the mean survival times estimated using the Fuzzy-PLE, the PLE and the vague uncertainty measure of S. Again it is seen that from times 60 to 100 the estimates tend to converge.

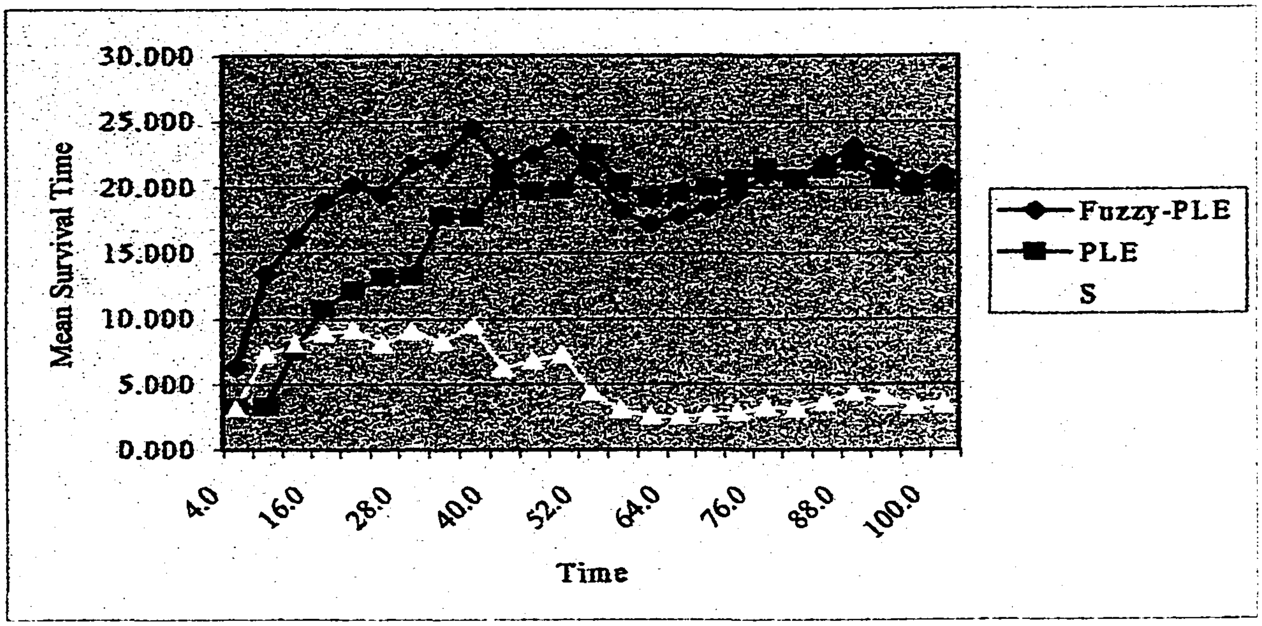


Figure 6.4 Plots of the Mean Survival Times

Now observe how the amount of vague uncertainty changes as more information becomes available. As seen in Table 6.4, S jumps from 3.186 at time 4 to 7.212 at time 8. The amount of vague uncertainty in the estimates, as measured by S, remains high (above 6.1) until time 52. At this time the proportion of failures in the data set is 2/3. After this time the value of S does not exceed 4.4. Looking at Table 6.4 the 90% confidence interval for S has the same patterns. Between time 8 and 48 the lower bound of the confidence intervals are not less than 2.2 and the upper bounds are not less than 9.9. After time 48, none of the lower bounds exceed 2.1 and only two of the upper bounds exceed 9.9. In general this trend indicates that, in this example, the Fuzzy-PLE is needed to obtain accurate estimates of the mean survival time up through time 48 and after that time the amount of vague uncertainty quantified by the method is significantly reduced. This is illustrated in Figure 6.4. Early there is much vague uncertainty as indicated by the values of S plotted with triangles. Going to the left on the time axis it is seen that the values of S stabilize after time 60. There is little vague uncertainty relative to the estimate after time 60. The vague uncertainty will asymptotically approach zero as time goes on and more data becomes available. It reaches zero when no censored values are used in the estimate.

6.2.1 Historical Data to Smooth Estimate Trends

The Fuzzy-PLE reaches a high estimate of 24.447 at time 36. The next estimate, at time 40, is lower and looking down the column several rises and dips occur in the estimates. Why does this happen? As an example, consider the data available at times 36 and 40, shown in Tables 6.5 and 6.6 respectively.

At time 36 there are twenty-four data points, and at time 40 there are twenty-six data points. Thus two of the censored units in Table 6.5 failed within the next four time units. The failures occurred on two of the units censored at time 36.0, resulting in the added failure times of 36.110 and 38.450, and the two censored times of 1.550 and 3.890. The other eight censored times were increased by four. Now in Table 6.6, instead of having the four largest times being censored, there are two censored and two failed times. These added data provide significantly less evidence of continued survival beyond time 40. In addition, two very early censored times are added to the data set. In Table 6.5 eight of the ten censored values are greater than 11.0 (also recognizing that the eight are in the upper half of the data set), and in Table 6.6 only six of the censored values are in the upper half of the data set, greater than 11.0.

The failure of several large censored values in the data set then can have a big effect on the estimates, not only in the loss of optimism of continued survival but also in the addition of small censored values into the data set. This is all evidence of an earlier mean survival time of 21.583 than the estimate of 24.447 made at time 36. Each time a decrease in the estimate of the mean survival time using the Fuzzy-PLE is observed in Table 6.4 the cause is a similar circumstance to what is described here, that is, several late censored units experiencing failures.

Table 6.5 Data Set at Time = 36

t_i	d_i
0.350	1.0
1.009	0.0
2.472	0.0
2.691	0.0
3.505	0.0
4.340	1.0
6.339	0.0
6.761	0.0
6.959	0.0
9.170	0.0
9.197	0.0
10.522	0.0
12.918	1.0
13.910	0.0
15.131	1.0
16.614	0.0
18.377	1.0
20.610	0.0
26.156	1.0
31.660	0.0
33.309	1.0
36.000	1.0
36.000	1.0
36.000	1.0

Table 6.6 Data Set at Time = 40

t_i	d_i
1.009	0.0
1.550	1.0
2.472	0.0
2.691	0.0
3.505	0.0
3.890	1.0
4.350	1.0
6.339	0.0
6.761	0.0
6.959	0.0
8.340	1.0
9.170	0.0
9.197	0.0
10.522	0.0
13.910	0.0
16.614	0.0
16.918	1.0
19.131	1.0
20.610	0.0
22.377	1.0
30.156	1.0
31.660	0.0
36.110	0.0
37.309	1.0
38.450	0
40.000	1

The effect of early censor times is unfairly reducing the estimate in these cases. One solution is to not give these early censored times as much weight in the estimate and to give more weight to longer surviving censored units and failure times. Using weighted averages of the current and previous estimates provides the desired result. Table 6.7 provides results using the several weights with the data in Table 6.1. The first two rows have not been modified from the original estimate. The estimates after time eight are weighted. The headings in the columns indicate the proportion of weights. The third column is labeled *1, 3* which indicates that the estimate in the i^{th} row ($i > 2$) is one part (0.25) the estimate of the mean survival time calculated using the Fuzzy-PLE at the current time and three parts (0.75) the immediate previous result. The fourth column is labeled *3, 1* indicating three parts the current estimate and one part the previous result. The fifth column is proportioned as 7 parts the current estimate, 20% the immediate predecessor, and 10% from the second earlier result. The last column uses the current result and the preceding results up to the four preceding. Thus, the estimate made at time twelve has proportions of 2/3 and 1/3. The estimate at time sixteen has proportions of 3/6, 2/6, and 1/6. The estimate at time twenty uses the proportions of 4/10, 3/10, 2/10, and 1/10. After time twenty the estimates are made using the proportions of 5/15, 4/15, 3/15, 2/15, and 1/15. This method of weighting allows the estimates to be made putting more weight on older data and less on the small censored times.

Each of the techniques helps to smooth the consecutive estimates. When stronger weights are put on the previous estimates there are smaller differences between consecutive estimates. And with larger weights on the current estimate there are larger differences or less of a smoothing effect. The first method (column three) has the most effect in this regard, but it holds the estimates low before time forty, because it is strongly weighted to the previous estimate. The next most effective is the last method, but it also has the problem of low estimates early on. This problem occurs in all of these cases to some degree. A solution is to understand the motivation of using these weights.

Table 6.7 Weighted Averages

Time	$\tilde{S}$	*1, 3*	*3, 1*	*7, 2, 1*	*H=5*
4	6.263	6.263	6.263	6.263	6.263
8	13.366	13.366	13.366	13.366	13.366
12	16.010	14.027	15.349	14.507	15.129
16	18.852	15.233	17.976	17.434	16.697
20	20.165	16.466	19.618	19.053	17.437
24	19.416	17.204	19.467	19.145	17.850
28	21.707	18.330	21.147	20.929	19.195
32	22.095	19.271	21.858	21.567	19.910
36	24.447	20.565	23.800	23.519	21.376
40	21.583	20.819	22.137	21.969	20.947
44	22.444	21.226	22.367	22.456	21.528
48	23.773	21.862	23.422	23.329	22.295
52	21.102	21.672	21.682	21.683	21.530
56	18.153	20.793	19.035	19.377	20.332
60	17.170	19.887	17.636	18.063	19.503
64	17.808	19.367	17.765	18.016	19.193
68	18.442	19.136	18.273	18.319	19.069
72	19.573	19.245	19.248	19.166	19.339
76	20.879	19.654	20.471	20.281	19.886
80	20.488	19.862	20.484	20.314	19.936
84	21.837	20.356	21.499	21.377	20.627
88	23.045	21.028	22.658	22.438	21.382
92	21.674	21.190	21.920	21.797	21.203
96	20.496	21.016	20.852	20.951	20.898
100	20.957	21.001	20.931	21.040	21.031

The problem with the method is that on the early estimates when there is very little information and it is desired to rely more on the fuzzy membership functions, the results are weighted too much on the early times. This result is then propagated through later estimates. The problem being addressed of having large censored times fail adding failure times and small censored times to the data set for the next estimate did not occur until time forty. Thus, Table 6.8 shows the results using the same techniques as in Table 6.7 except that before time forty no weighting is used. All of the estimates made starting at time forty use the weights. In these cases, the estimates use the strong influence of the fuzzy membership functions in the early estimates and this information is then propagated through the rest of the estimates, positively effecting the results. It is seen in both tables that after time sixty-four the weighting techniques do not have much of an effect on the estimates. In these cases there are sufficient data to dictate the result.

Table 6.8 Weighted Averages Used After Time 40

Time	$\tilde{S}$	*1, 3*	*3, 1*	*7, 2, 1*	*n=5*
4	6.263	6.263	6.263	6.263	6.263
8	13.366	13.366	13.366	13.366	13.366
12	16.010	16.010	16.010	16.010	16.010
16	18.852	18.852	18.852	18.852	18.852
20	20.165	20.165	20.165	20.165	20.165
24	19.416	19.416	19.416	19.416	19.416
28	21.707	21.707	21.707	21.707	21.707
32	22.095	22.095	22.095	22.095	22.095
36	24.447	24.447	24.447	24.447	24.447
40	21.583	23.731	22.299	22.207	22.557
44	22.444	23.409	22.408	22.597	22.843
48	23.773	23.500	23.432	23.381	23.318
52	21.102	22.901	21.684	21.707	22.261
56	18.153	21.714	19.036	19.387	20.888
60	17.170	20.578	17.637	18.067	19.918
64	17.808	19.885	17.765	18.018	19.502
68	18.442	19.525	18.273	18.320	19.300
72	19.573	19.537	19.248	19.167	19.511
76	20.879	19.872	20.471	20.281	20.015
80	20.488	20.026	20.484	20.314	20.032
84	21.837	20.479	21.499	21.377	20.699
88	23.045	21.120	22.658	22.438	21.436
92	21.674	21.259	21.920	21.797	21.243
96	20.496	21.068	20.852	20.951	20.928
100	20.957	21.040	20.931	21.040	21.054

To study the estimate of the mean survival time using the Fuzzy-PLE, this issue will not be considered in the remainder of this chapter. It is suggested to use this technique in the analysis of data when the above situation arises, but it can confound the analysis of the efficacy of using the Fuzzy-PLE.

In conclusion, this section provides a simulation that involves the tracking of in-service equipment over time and investigates the behavior of the estimates of the mean survival time using the Fuzzy-PLE and the PLE. In addition, some indicators to determine at which point sufficient data may exist to use the PLE instead of the Fuzzy-PLE are given. The three indicators that are discussed are

(1) The observance that the estimates made using the Fuzzy-PLE and PLE agreed,

(2) the decrease in the amount of vague uncertainty quantified in the Fuzzy-PLE estimate,

(3) that the percentage of censored values in the data set dropped to below 30%.

One of these factors alone may not be conclusive evidence to use only the standard statistical method but, combined they are a strong indicator that enough data exists.

As stated in Chapter 3, there exist three parameters that are user input to the method. The next section considers the effects of these parameters using the data from the simulation of this section and comparisons are made with the results from this section.

6.3 The Effects of User Input on the Estimates

This section considers the effects that the user input of expert knowledge (*ek*), the associated confidence (*uc*), and the optimism parameter U have on the estimates. The data set used in the previous section is used and estimates are made with several different values of these parameters. Comparisons are made with the results from the last section, the PLE, and between results obtained in this section. In addition, discussions of the different parameters are made in an attempt to obtain an understanding of the proper use of the given parameters.

U is a parameter that indicates the user's optimism of continued survival of the censored units. The parameter can vary between 50 and 100. As stated previously, all of the simulations performed up to now have been made with $U = 0.65$. This is a "middle" value for the optimism and the value of U used in the development in Chapter 3. The other two parameters involve the user's belief about the actual survival time of the censored units. The user can provide expert knowledge (*ek*) which represents a belief of the survival time of the censored units. Associated with the expert knowledge the user inputs a confidence (*uc*) in the expert knowledge (*ek*). The difference in the two types of input is that with U the user is providing an input about the amount of optimism in continued survival of the censored units (i.e., the censored units will survive not much longer, longer, very much longer, etc...). With the *ek* and *uc* input, the user is providing information in the belief (expert knowledge) of the survival time for the equipment in which the estimates are being made (i.e. 85% confident that the survival time for this type of equipment is 26.8 months). All estimates made until now have been made with no such input.

6.3.1 Expert Knowledge of the Survival Time

The effects of changing *ek* and *uc* are considered in the current subsection. *ek* is the expert knowledge of the survival time for the units and *uc* is the amount of confidence in the expert knowledge on a scale from 0 to 100. For example, the user of the method may believe that the survival time of the units in question may be 48.3. Thus this is the value that the parameter *ek* is given. And the user may have a strong opinion about this belief. Thus *uc* could be set to 87 to represent a high level of confidence in this knowledge.

The expert knowledge could be based on considering "like" equipment. Then the value of *ek* may be close to the survival time of the "like" equipment, and the value of *uc* is based on the amount of "likeness" between the two types of equipment. For example, there may exist much historical data on an earlier version of an engine and an estimate is made on the mean survival time for this equipment. Then it is determined that the new version of the engine is "very similar" to the old equipment. Thus a large value for the confidence (*uc*) is given, and *ek* is set to the mean survival time calculated using the old equipment's data. These values of *ek* and *uc* are then used in the method when making estimates on the new equipment. *ek* and *uc* would continue to be used in the estimates until enough data from the new equipment exists to make accurate estimates without the data from the old equipment. Another acquisition of expert knowledge could be an actual expert's opinion and then *uc* is truly the expert's confidence in the opinion. Another scenario that may arise is that old data exist for the current equipment, but the old data are questionable. It may be that the old data had been collected inaccurately or using a procedure deemed not reliable at a latter time. Now the system has been corrected and the new data is considered "clean," but there are very few data, and most is censored. Thus an estimate made from the old data could be used as *ek* and *uc* is set according to how much weight is to be put on the old data estimates. In this case, as more "clean" data becomes available the value of *uc* may be decreased proportionally, putting more weight on the "clean" data and less on the old data.

The data used in Section 6.2 is from an *exponential*(24) distribution. Suppose the expert believes very strongly that the actual survival time of the units is 24. Thus *ek* is set to 24 and *uc* is set to 90. Again U is left at 0.65 as it was in the previous section. The results of the simulation using this information are shown in Table 6.9. The header of the third column is subscripted with *ek* indicating that the estimates are made with $ek = 24$ and $uc = 90$. For a comparison the fourth column is the estimate made using the Fuzzy-PLE with *ek* and *uc* both set to zero. As may be expected the estimate made at time 4 of 17.707 is very close to the true mean of 24 compared to the estimate of 6.263 made with no expert knowledge. Also the estimate made at time 8 of 18.358 is an improvement over the estimate of 13.366 made previously using the Fuzzy-PLE with no expert knowledge input.

The minimum estimate of 16.270 occurs at time 24 and the maximum estimate of 20.538 is made at time 88. This small variation of the estimates demonstrates that with this accurate expert knowledge the estimates are very consistent across time. In addition, the vague uncertainty as measured with S (in this case S_{ek}) starts high and steadily decreases, as more data becomes available. This combination is indicative of the incorporation of accurate expert knowledge into the method.

With the exception of times 56, 60, and 64, after time 16 the estimates are consistently lower than the estimates made with no expert knowledge. In addition, the confidence intervals only cover the mean of the distribution (24) in six of the estimates, compared to sixteen of the confidence intervals covering the mean of 24 when no expert knowledge is incorporated into the estimate. The expert knowledge of 24 is correct. Why then are the estimates less accurate? The answer has to do with what *ek* represents and how it is used in the method.

Table 6.9 Results of the Simulation with $ek = 24$ and $uc = 90$

Time	*n*	$\tilde{S}_{ek}$	$\tilde{S}$	S_{ek}	$se_B(\tilde{S}_{ek})$	CI($\tilde{S}_{ek}$)	CI(S_{ek})
4	13	17.707	6.263	14.630	0.1371	(12.9857, 20.8956)	(10.0657, 17.6793)
8	13	18.358	13.366	12.204	0.1481	(13.3490, 21.7187)	(8.4070, 14.8379)
12	15	17.269	16.010	9.268	0.1863	(12.8207, 21.0667)	(5.7655, 12.0812)
16	16	17.363	18.852	7.362	0.2035	(13.3103, 21.0811)	(4.6406, 10.3013)
20	18	17.087	20.165	5.975	0.1665	(13.3432, 20.6239)	(3.5466, 9.0725)
24	21	16.270	19.416	4.841	0.2093	(12.7409, 20.0530)	(2.5848, 7.7615)
28	22	16.836	21.707	4.108	0.2476	(12.9113, 21.1838)	(2.2499, 6.8552)
32	23	17.957	22.095	4.043	0.2047	(13.9521, 22.2945)	(2.1563, 6.5126)
36	24	18.619	24.447	3.619	0.2753	(14.5855, 23.6613)	(1.8183, 6.6718)
40	26	19.270	21.583	3.885	0.2686	(15.1600, 23.5520)	(1.9452, 6.7058)
44	28	19.579	22.444	3.864	0.2373	(15.7643, 24.1802)	(2.0966, 6.5170)
48	29	19.823	23.773	3.271	0.2923	(15.7474, 24.6701)	(1.7155, 5.5598)
52	31	20.309	21.102	3.534	0.2351	(16.4930, 24.7621)	(1.8585, 5.7653)
56	37	18.651	18.153	3.515	0.2112	(15.1003, 22.8883)	(1.9638, 5.7335)
60	41	18.441	17.170	3.806	0.1990	(15.0771, 22.5737)	(2.0460, 5.9183)
64	42	18.341	17.808	3.102	0.2027	(15.0702, 22.3152)	(1.6481, 4.9764)
68	43	18.347	18.442	2.533	0.1618	(14.8317, 22.0164)	(1.2591, 4.1954)
72	43	18.706	19.573	1.961	0.2204	(15.1420, 22.4237)	(1.0345, 3.3339)
76	43	19.080	20.879	1.405	0.2294	(15.8788, 23.7420)	(0.6651, 2.3672)
80	46	19.490	20.488	2.098	0.1632	(16.2183, 22.8930)	(0.9826, 3.7150)
84	46	19.939	21.837	1.678	0.1805	(16.6142, 23.9049)	(0.7497, 3.0193)
88	47	20.538	23.045	1.814	0.1632	(17.0012, 24.6596)	(0.9272, 3.4466)
92	52	20.211	21.674	2.518	0.1703	(16.8864, 24.1731)	(1.4543, 4.3812)
96	56	19.920	20.496	2.777	0.1717	(16.8648, 23.8238)	(1.6779, 4.7779)
100	57	19.829	20.957	2.285	0.1906	(17.1268, 24.2074)	(1.4006, 4.0179)

The justifications and mathematical derivations for the incorporation of the expert knowledge (*ek*) and confidence (*uc*) are given in Chapter 3. The expert knowledge provides an exact value ("exact" depending on the value of the associated confidence *uc*) for the survival time. This is not the same as the mean survival time. Given a "large" value of *uc* the individual censored times will approach a step function. In this example, *ek* is 24 and *uc* is 90. As stated in chapter 3, a large value of *uc* causes censored values less than or equal to 24 to approach a step function with the step at time 24 and censored values greater than 24 to approach a step function about the censor time. This eliminates most of the optimism of continued survival beyond *ek* and *the* censored time, respectively.

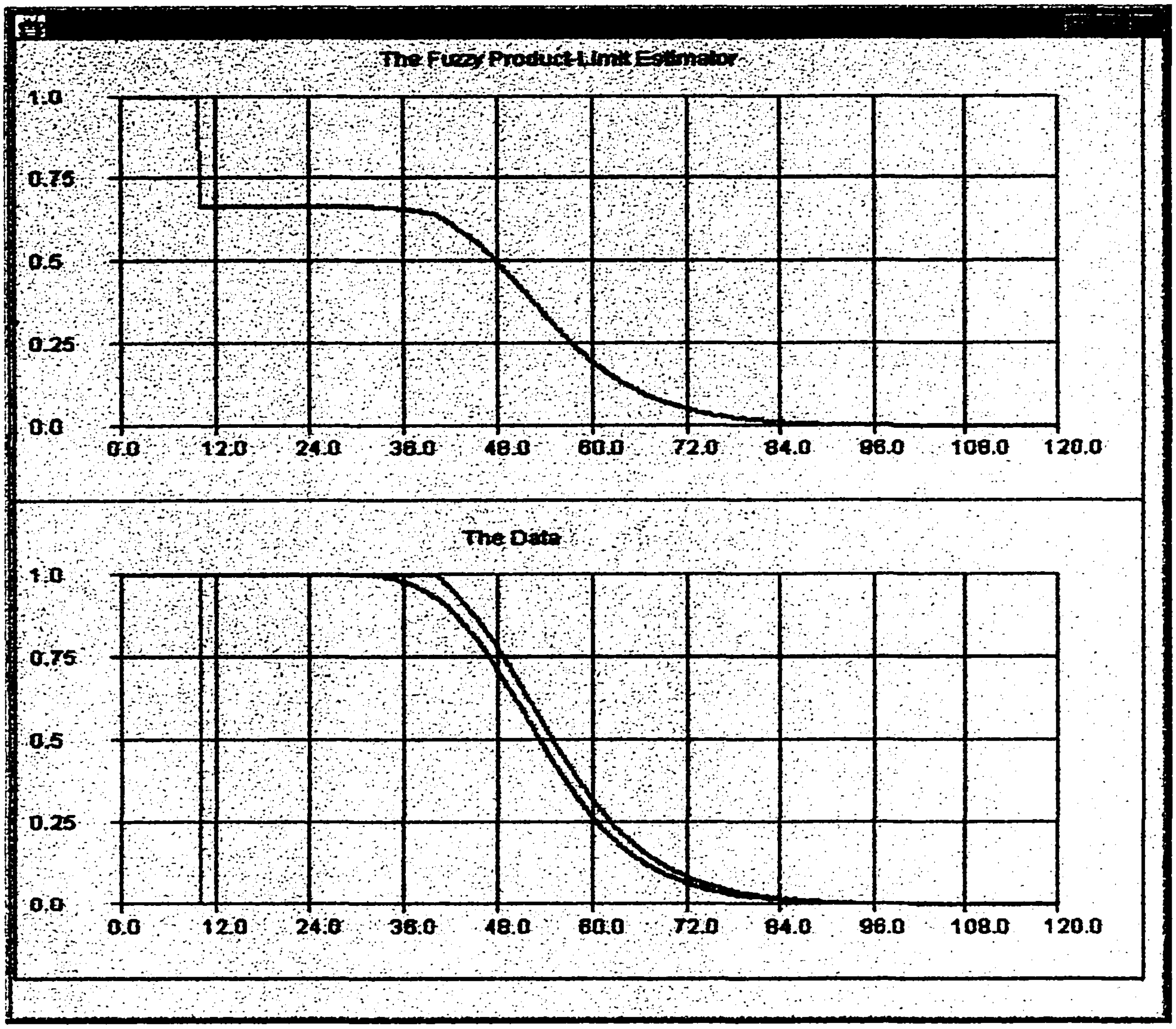


Figure 6.5 Three Data Points with $ek = 0$ and $uc = 0$

Consider an example with the data set {(10, 0), (30, 1), (40, 1)}. Without any expert knowledge (ek = 0 and uc = 0) incorporated into the estimate the mean survival time calculated using the Fuzzy-PLE is 40.102. Figure 6.5 shows a graph of the survival curve in the top half of the figure and graphs of the membership functions in the bottom half of the figure when no expert knowledge is used.

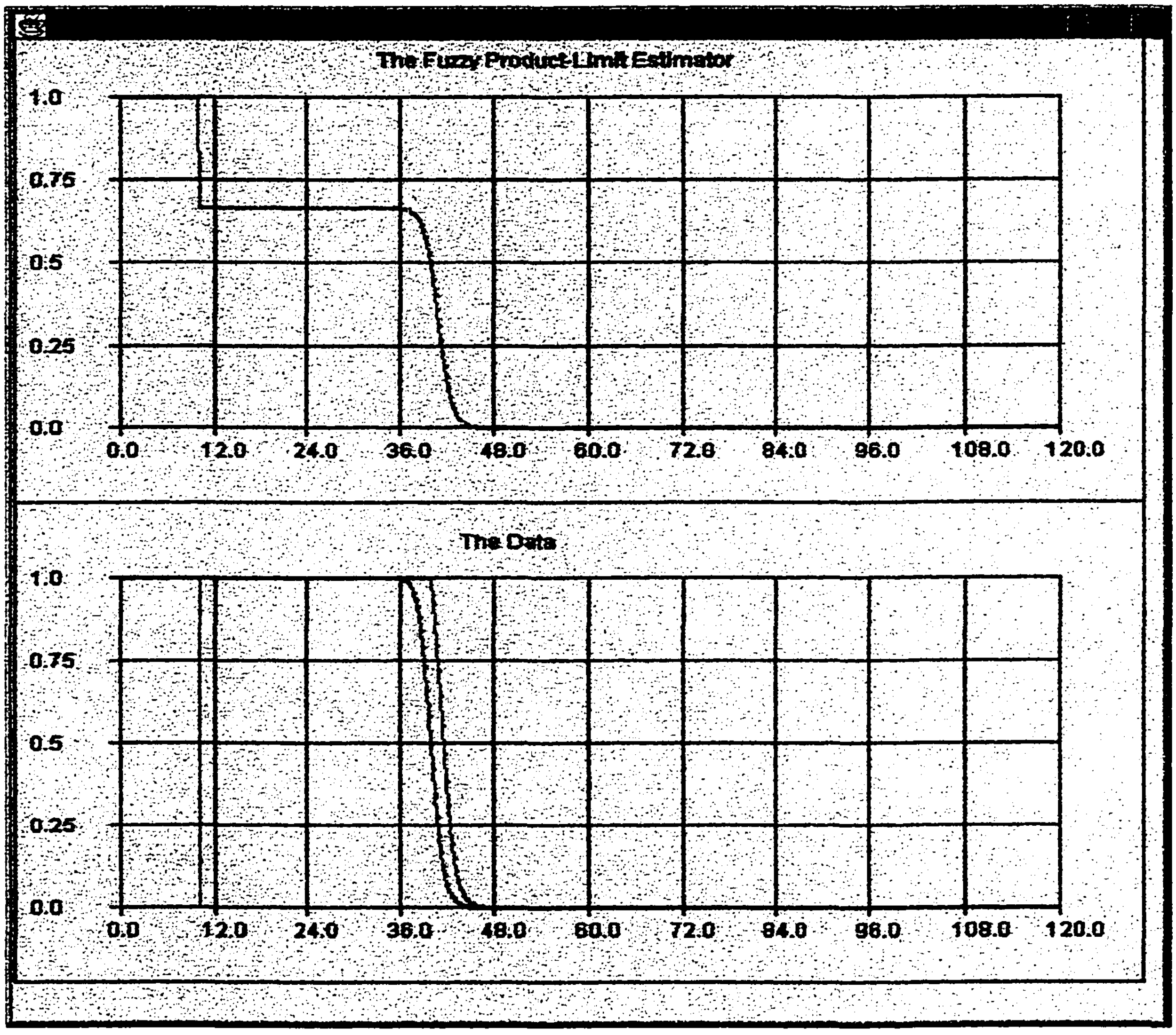


Figure 6.6 Three Data Points with ek = 50 and uc = 90

Figure 6.6 shows the graphs with *ek* set to 38 and *uc* set to 90. The mean survival time estimated with this added information is 30.546. Comparing the graphs of the survival curve in Figure 6.5 with that in Figure 6.6 shows that both drop to 0.667 after the failure at time 10 and remain the same up to time 38. After that time they differ. The graph in Figure 6.6 drops off sharply because the incorporated expert knowledge forces the censored values to step to zero at times 38 and 40. When no expert knowledge is incorporated the membership functions for these censored times are allowed to slope off at a slower rate. Thus in this example the expert knowledge of 38 causes the estimate to be further from the expert knowledge rather than closer. The reason is that the data contain the early failure time of 10, and the two censored times only account for 67% of the data. Forcing the censored times to approach step functions at the times 38 and 40 causes the data set to approach a data set containing three failure times at 10, 38, and 40. Thus diminishing the belief of continued survival of the censored units. This is a specific scenario that helps to explain why the correct expert knowledge about survival time can have an adverse effect on the resulting estimate of the mean survival time, as is seen in the example of the *exponential*(24) data. The problem is that of the early failure time and early failure times occur with *exponential* distributions.

If the confidence in the expert knowledge is lower then the censored data will have membership functions less like step functions. Table 6.10 shows the estimates of the mean survival time made using the Fuzzy-PLE with the data from section 6.2. In all of the estimates *ek* is 24 and *uc* increases across the columns.

Considering each column it is seen that as *uc* increases, prior to time 16 the estimates increase because the censoring times are small and the membership functions are stretched, increasing to time 24. From time 16 through 48 the opposite effect occurs and the estimates decrease as *uc* increases. For the times 56, 60, and 64 the estimates again increase as *uc* increases. There is very little difference as *uc* increases for the times 52 and 68. This lack of difference is caused by early censoring times in the data sets.

The difference between the smallest and largest in each column decreases as *uc* increases. When no expert knowledge is incorporated (*uc* = 0) the difference is 18.184. As *uc* increases from 10 to 90 the differences are 14.323, 12.794, 9.332, 6.107, and 4.268 respectively. The difference between the smallest and largest estimate of the mean survival time using the PLE as seen in Table 6.4 is the largest, at 19.341. Thus an effect of incorporating this expert knowledge is that the estimates are more consistent across time as it should be if the correct answer is given some weight in the estimates.

The benefit of using *ek* is that when few data are available, more accurate estimates can be obtained and the estimates remain more consistent as more data becomes available. The disadvantage is that in cases like with exponentially distributed data, the early failures can cause underestimates because *ek* provides information about the survival time of an individual unit, not the mean survival time, and with a large variance in the data set *ek* is a hindrance to the actual failure times. This hindrance occurs because the estimates are intentionally made early and the large variance implies that some failures will be observed earlier than the mean survival time.

Now consider what happens if inaccurate expert knowledge is incorporated into the method. In the following simulations, the same data will be used with *ek* set to 12, 36, and 48. For each, the very high level of confidence will be given by setting *uc* equal to 90 as was done previously.

Table 6.10 The Effects of Varying *uc*

Time	$uc = 0$	$uc = 10$	$uc = 30$	$uc = 50$	$uc = 70$	$uc = 90$
4	6.263	7.342	9.795	12.410	15.062	17.707
8	13.366	13.721	14.604	15.726	17.011	18.358
12	16.010	16.057	16.184	16.415	16.782	17.269
16	18.852	18.660	18.271	17.900	17.577	17.363
20	20.165	19.846	19.188	18.503	17.790	17.087
24	19.416	19.100	18.455	17.782	17.062	16.270
28	21.707	21.193	20.154	19.093	17.994	16.836
32	22.095	21.665	20.794	19.899	18.963	17.957
36	24.447	23.834	22.589	21.316	20.001	18.619
40	21.583	21.362	20.906	20.419	19.883	19.270
44	22.444	22.165	21.591	20.983	20.322	19.579
48	23.773	23.368	22.545	21.693	20.796	19.823
52	21.102	21.049	20.930	20.783	20.587	20.309
56	18.153	18.243	18.411	18.549	18.640	18.651
60	17.170	17.351	17.697	18.010	18.268	18.441
64	17.808	17.901	18.076	18.222	18.320	18.341
68	18.442	18.460	18.486	18.489	18.452	18.347
72	19.573	19.501	19.348	19.176	18.970	18.706
76	20.879	20.703	20.342	19.961	19.547	19.080
80	20.488	20.408	20.235	20.036	19.795	19.490
84	21.837	21.651	21.269	20.866	20.429	19.939
88	23.045	22.794	22.280	21.742	21.169	20.538
92	21.674	21.547	21.276	20.977	20.630	20.211
96	20.496	20.469	20.398	20.296	20.146	19.920
100	20.957	20.865	20.668	20.442	20.172	19.829

First, consider the case of the expert knowledge being very low relative to the actual mean of the distribution. In Table 6.11 the estimates are made with the incorporation of the expert knowledge set at 12, which is half the actual mean of the distribution. In this case the estimate of the mean survival time calculated using the Fuzzy-PLE is low during the beginning times and gradually increases. Also, the vague uncertainty measured by S_{ek} is 6.321 in the estimate made at time 4 but quickly drops to 3.903 at time 8 and 2.215 at time 12. S_{ek} is then measured between 2.373 and 0.428 for the rest of the estimates. Consider these values in contrast to the estimates taken with an accurate ek in which $\widetilde{S}_{ek}$ are consistent for the entire time period and S_{ek} started very large and gradually decreased.

Table 6.11 Results of the Simulation with ek = 12 and uc = 90

Time	*n*	$\widetilde{S}_{ek}$	$\widetilde{S}$	S_{ek}	$se_B(\widetilde{S}_{ek})$	CI($\widetilde{S}_{ek}$)	CI(S_{ek})
4	13	9.398	6.263	6.321	0.068	(6.9912, 10.8857)	(4.3107, 7.7593)
8	13	10.057	13.366	3.903	0.0761	(7.5611, 11.7531)	(2.4296, 4.8841)
12	15	10.216	16.010	2.215	0.0963	(8.0975, 12.0062)	(1.0702, 3.4924)
16	16	12.126	18.852	2.125	0.1707	(9.6705, 14.7199)	(1.0649, 3.3384)
20	18	13.485	20.165	2.373	0.1922	(10.7658, 16.6381)	(1.1881, 3.7133)
24	21	13.679	19.416	2.250	0.1770	(10.6326, 17.0030)	(1.1724, 3.5378)
28	22	14.641	21.707	1.913	0.2428	(11.0009, 18.9704)	(0.9211, 3.0158)
32	23	15.630	22.095	1.716	0.1991	(11.7581, 19.8907)	(0.7875, 2.8257)
36	24	16.747	24.447	1.747	0.2513	(12.4277, 21.3786)	(0.7091, 3.0146)
40	26	17.124	21.583	1.739	0.2155	(13.3553, 21.9905)	(0.8078, 2.861)
44	28	17.550	22.444	1.835	0.2683	(13.7724, 22.7534)	(1.0291, 3.0045)
48	29	17.963	23.773	1.411	0.2586	(14.2164, 23.3176)	(0.6998, 2.4743)
52	31	18.092	21.102	1.317	0.2082	(14.4624, 22.8387)	(0.6310, 2.3861)
56	37	16.518	18.153	1.382	0.1874	(13.2994, 20.7862)	(0.6226, 2.4925)
60	41	16.307	17.170	1.672	0.2101	(12.8800, 20.3627)	(0.9158, 2.6809)
64	42	16.438	17.808	1.199	0.1578	(13.1094, 20.1588)	(0.6783, 1.9355)
68	43	16.612	18.442	0.798	0.1742	(13.6138, 20.5640)	(0.3755,1.4899)
72	43	17.274	19.573	0.529	0.1954	(13.9864, 20.9874)	(0.2473, 0.9779)
76	43	18.103	20.879	0.428	0.2320	(14.5923, 21.8807)	(0.2185, 0.7841)
80	46	18.355	20.488	0.963	0.1433	(15.2137, 22.0949)	(0.4144, 1.7814)
84	46	19.036	21.837	0.775	0.1833	(15.5304, 22.9156)	(0.3462, 1.4316)
88	47	19.549	23.045	0.825	0.1934	(16.2840, 24.0049)	(0.4358, 1.6686)
92	52	18.834	21.674	1.141	0.2026	(15.5192, 23.0142)	(0.5839, 2.0362)
96	56	18.454	20.496	1.311	0.1885	(15.3111, 22.3239)	(0.7469, 2.2138)
100	57	18.454	20.957	0.910	0.1737	(15.4935, 22.6176)	(0.5217, 1.6734)

Notice that after time 60 the estimates of mean survival time made with ek = 12 and ek = 24 are within two of each other. After time 76 this small difference is true between the estimates made with no expert knowledge and these estimates. Also the upper and lower bounds of the confidence intervals made with $ek = 12$ and $ek = 24$ are very close in the latter estimates. These latter estimates demonstrate that when enough data is available the expert knowledge yields to the recorded data.

Now consider the case of expert knowledge that is wrong with a much larger value of the survival time than the mean of the distribution. First, the extreme case of twice the actual mean ($ek = 48$) is considered. The screenshot of the results is shown in Figure 6.7. The graphs in the top left show the estimates of the survival curves and the bottom graph is the confidence interval for the last estimate made at time 100. The graphs of the survival curves are color coded with the times in the table on the right side of the screen shot. The survival curves are generated at the same times as the survival curves in the original simulation from Section 6.2 so that a direct comparison can be made with the survival curves of Figure 6.1. The survival curve made at time 4 decreases early because of the failures; then because there are ten censored values, it continues out to about time forty-two. Then it takes a sharp turn to zero at time 46. Looking at the other curves it is seen that the early curves generally continue above the curve estimated at time 100 prior to time 44 because the earlier curves are estimated with fewer data and are more influenced by the expert knowledge than the later. This result is what is expected and desired; that is, before sufficient data are available, the estimate is heavily influenced by the expert knowledge, and as more data become available, less weight is on the expert knowledge.

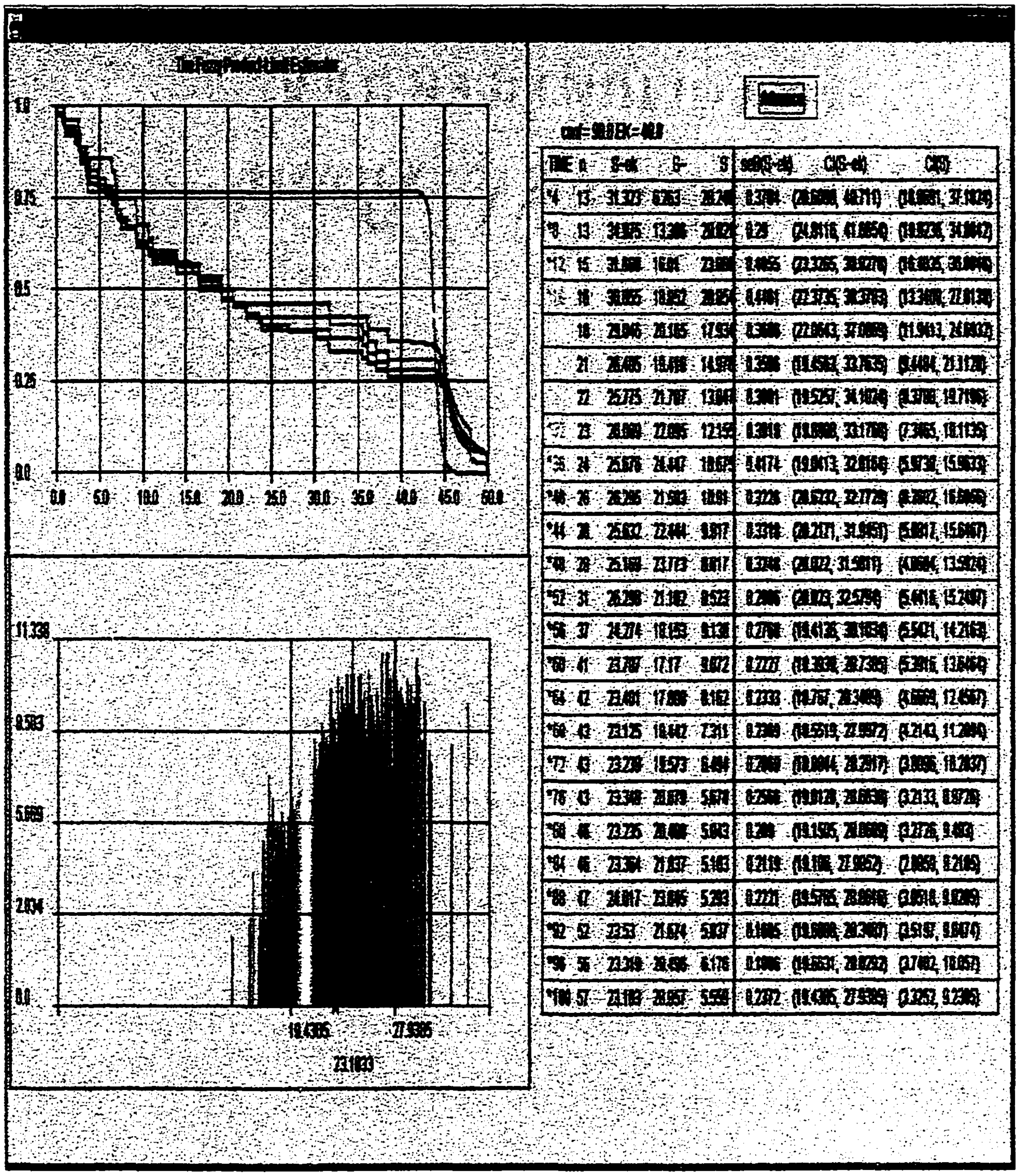


Figure 6.7 Results of the Simulation with $ek = 48$ and $uc = 90$

The estimates of the mean survival time start very high (in the 30s) and drop rapidly as more data becomes available. This drop is an indication that the expert knowledge is incorrect (too large). A good estimate of 26.405 is obtained at time 24, the estimates slowly decrease, and this simulation ends with the low estimate of 23.103 at time 100. It is noted that all of the confidence intervals contain the mean of the distribution except for the estimate made at time 8. Again it is seen that the data overcome this incorrect input, and the estimates approach the "correct" solution as sufficient data become available.

The next simulation is performed with a more moderate "overestimate" from the expert. The expert knowledge of $ek = 36$ is used. Figure 6.8 shows the screenshot from this simulation. Here notice that the survival curves remain high until time 33 due to the influence of the expert knowledge, and then decrease.

Again notice that the estimates of the mean survival time start large and decrease, as more data becomes available. The decrease is more subtle in this case than in the case with $ek = 48$. As stated previously, this is an indication that the expert knowledge is too large of input although the more subtle change indicates it is closer to the actual mean of the data. In both cases the largest and smallest estimates are made at the times 8 and 68 respectively. The difference between the largest and smallest in this case is 6.053 compared to 11.850 in the previous case.

In this simulation all of the confidence intervals contain the mean of the distribution. In general, the widths of the confidence intervals are smaller in this simulation.

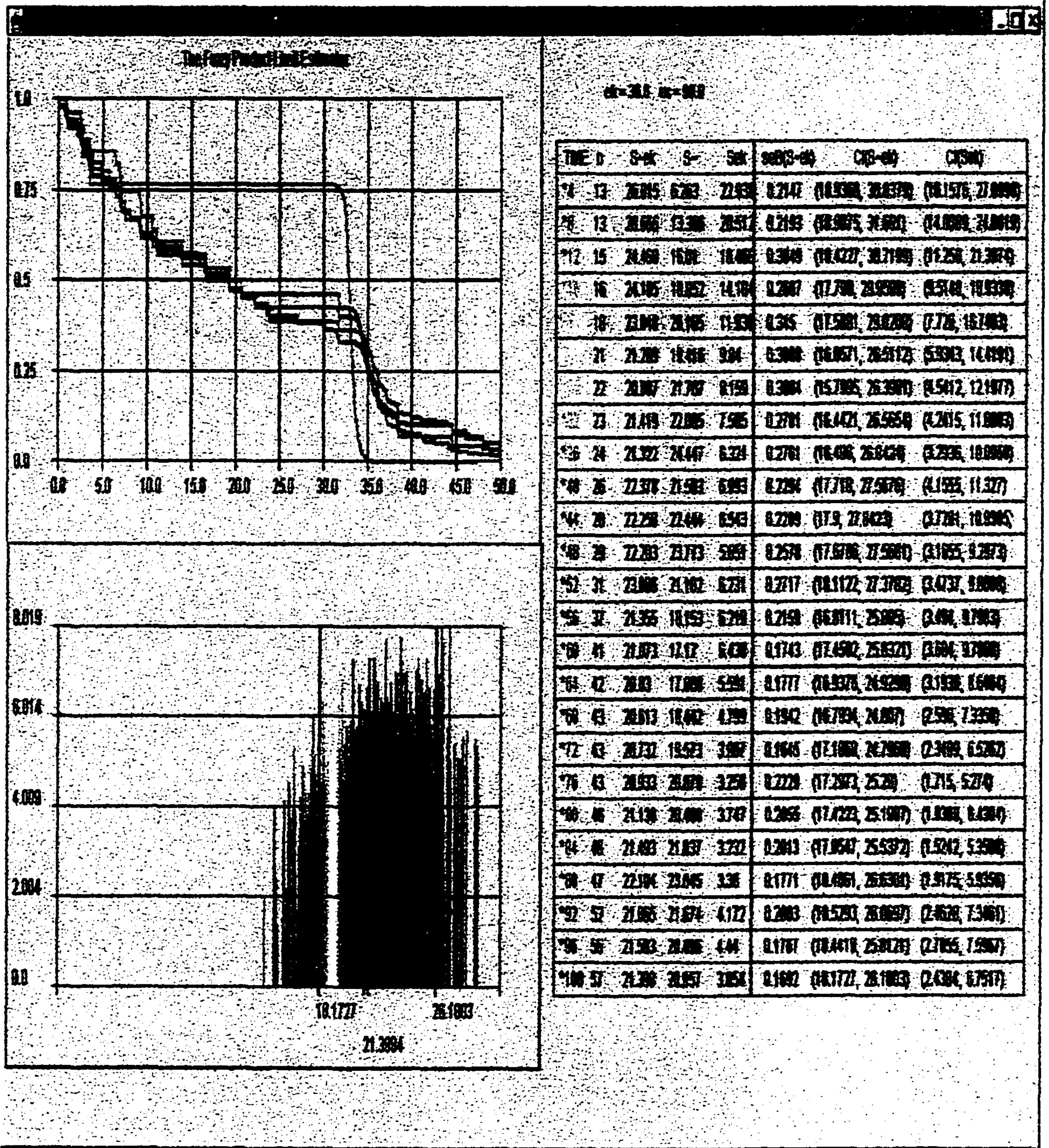


Figure 6.8 Results of the Simulation with $ek = 36$ and $uc = 90$

In this subsection the effects of incorporating expert knowledge about the survival time of the equipment is investigated. In the next subsection, the use of the optimism parameter U is investigated. All of the estimates so far have been made with this parameter set at 0.65. In the section, the two extremes of 0.51 and 0.99 are considered.

6.3.2 Optimism in Continued Survival of Censored Units

Now the effects of changing U are investigated. In the previous simulations U is set at 0.65. As discussed in Section 3.3.2, the minimum possible value of U is 0.51 and the maximum is 0.99. In this section the simulation over time is repeated using the same data that was used in section 6.2 but with $U = 0.51$ and then again with $U = 0.99$. In both cases no expert knowledge is used.

The motivation for changing the value of U may come from several places. A small value of U, say 0.51, may be used in a situation where conservative results are desired, or the user may have reasons to believe that the continued survival of the censored units may be limited. On the other hand, one may have reason to believe that the censored units will survive much longer. For instance, it may be known that the equipment will survive significantly longer than the censored times because the censoring is done very early. An example of this motivation is that data have been collected after two months of an engine model known to last several years. In such a situation there is much optimism about the continued survival of the censored units. As a rule of thumb, it is suggested to use the middle value of 0.65 when no strong opinion exists otherwise.

Considering the case of much optimism with U set equal to 0.99 the simulation is run and the results are shown in Table 6.12. It is seen that as early as time 4 a reasonable estimate of 9.625 is made as compared with the estimate of 6.263 made using the Fuzzy-PLE with U equal to 0.65 or the estimate of 3.222 made using the PLE. At time 8 an excellent estimate of 21.488 is made with a corresponding confidence interval of (12.413, 28.2205) which covers the mean of the distribution. This estimate is compared to the estimate made with the PLE of 3.25.

Table 6.12 Results of the Simulation with $U = 0.99$

Time	*n*	*P*	$\tilde{S}$	*S*	$seB(\tilde{S})$	CI($\tilde{S}$)	CI(*S*)
4	13	3.222	9.625	6.548	0.1972	(5.6689, 13.5543)	(3.1900, 9.9896)
8	13	3.250	21.488	15.334	0.3618	(12.4130, 28.2205)	(7.6064, 21.3125)
12	15	7.514	24.651	16.651	0.7181	(14.2213, 37.8773)	(7.9409, 28.3627)
16	16	10.720	28.219	18.219	0.7528	(16.3948, 42.6702)	(8.5113, 31.0237)
20	18	12.123	29.404	18.292	0.9504	(17.3032, 48.4301)	(8.5558, 35.0787)
24	21	13.160	27.332	15.903	1.1303	(16.6339, 46.7327)	(7.1842, 31.8681)
28	22	13.211	31.025	18.297	1.3042	(17.7874, 50.2612)	(7.8572, 35.6704)
32	23	17.810	30.501	16.587	1.5847	(18.2843, 51.7747)	(7.0547, 34.5747)
36	24	17.674	34.266	19.266	1.3488	(19.5909, 57.5714)	(7.7069, 39.0512)
40	26	20.433	27.608	12.223	1.4013	(18.5389, 43.8429)	(5.7015, 25.6030)
44	28	19.677	28.764	13.049	0.9614	(18.7675, 48.5889)	(6.1102, 29.6084)
48	29	19.763	30.935	14.383	1.1366	(19.2503, 52.0363)	(5.6536, 31.5944)
52	31	22.563	25.152	8.377	0.4751	(18.4289, 34.7153)	(4.1438, 15.5722)
56	37	20.347	20.695	5.559	0.3484	(15.7040, 26.8554)	(2.8994, 10.1348)
60	41	19.103	18.958	4.323	0.2124	(14.4997, 23.6917)	(2.4016, 7.2715)
64	42	19.564	19.909	4.670	0.2278	(15.4141, 25.0813)	(2.3952, 8.0575)
68	43	19.970	20.890	5.076	0.3180	(16.2435, 26.3085)	(2.7366, 8.8679)
72	43	20.609	22.477	5.732	0.3570	(17.3076, 28.3069)	(2.8699, 10.3854)
76	43	21.468	24.518	6.843	0.4375	(18.5825, 31.0537)	(3.3362, 12.6492)
80	46	20.634	23.264	6.232	0.5420	(18.0539, 31.4882)	(3.0260, 12.8502)
84	46	21.382	25.753	7.492	0.7830	(19.8041, 36.4980)	(3.3987, 16.2711)
88	47	21.864	27.141	8.417	0.6765	(20.7968, 40.2341)	(3.9125, 19.0297)
92	52	20.529	24.983	7.290	0.5935	(18.9050, 38.6260)	(3.1075, 18.4329)
96	56	19.939	22.963	5.820	0.4374	(18.1111, 36.3475)	(2.6024, 17.5328)
100	57	20.232	23.717	6.173	0.5556	(18.5770, 37.7861)	(2.7307, 18.0270)

Now considering the estimates as time increases. It appears that up through time 36 that the estimates are gradually increasing, getting overoptimistic and out of control. However, this high estimate of 34.266 seems quite large the confidence interval does cover the mean of 24, as do all of the confidence intervals beyond time 4. After time 36 the estimates generally decrease to between the upper teens and the lower twenties. The reason is that even though there is great optimism in the continued survival of the censored units, enough failure data have become available to keep the estimates near the mean of the distribution. It is noted that although after time 4 all of the confidence intervals cover the mean of the distribution, the intervals are significantly wider than in the previous simulations. Figure 6.9 shows the survival curves estimated using the Fuzzy-PLE for the times 4, 24, 44, 64, 84, and 100 as was shown previously. In this case the curves are generally more optimistic than the survival curves of Section 6.2. They do not have the strong influence at a given point, as was seen with the curves that incorporated expert knowledge in the estimates.

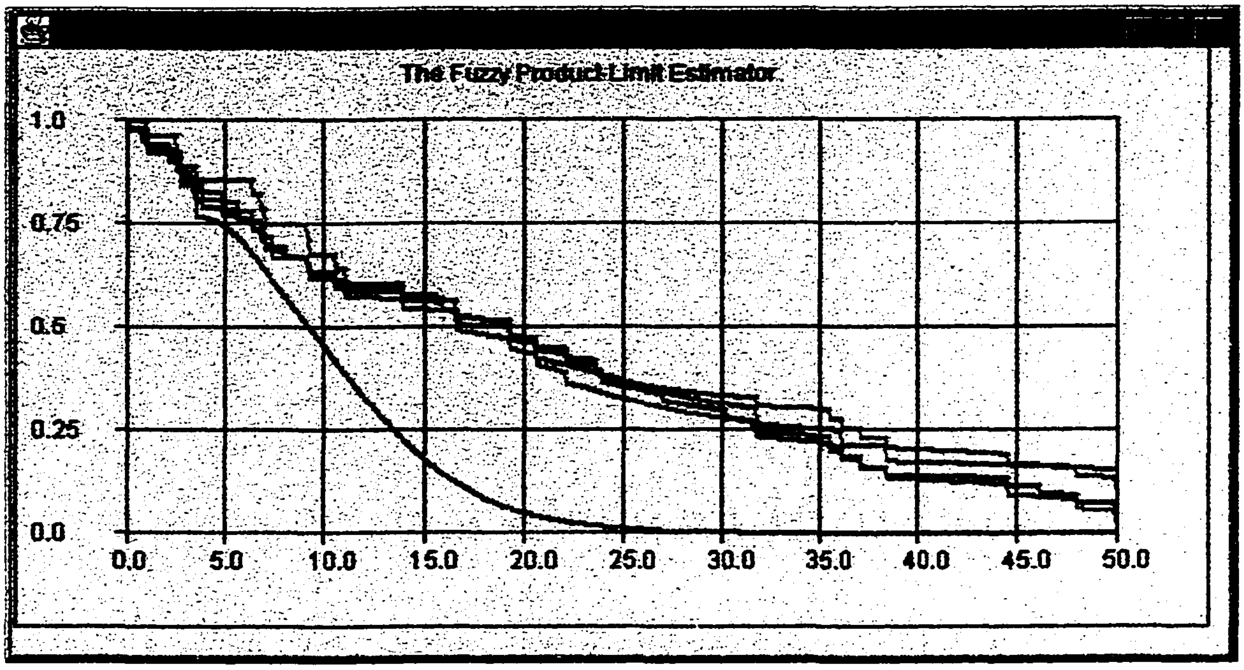


Figure 6.9 Survival Curves with $U = 0.99$

Now consider the other end of the spectrum, with $U = 0.51$. This is the case of very little optimism in the continued survival of the censored units. As may be expected, all of the estimates are lower than the estimates made with $U = 0.65$ and $U = 0.99$. Table 6.13 provides the results of the simulation. A comparison of the estimates made using the Fuzzy-PLE with those made using the PLE reveals that approximately the same pattern as that with $U = 0.65$ exists. Early estimates exceed those made with the PLE and the estimates made using the Fuzzy-PLE converge to the low twenties. The confidence intervals cover the mean of the distribution in sixteen of the cases.

Table 6.13 Results of the Simulation with $U = 0.51$

Time	*n*	*P*	$\widetilde{S}$	*S*	$seB(\widetilde{S})$	CI($\widetilde{S}$)	CI(*S*)
4	13	3.222	5.661	2.584	0.0798	(3.9339, 7.1953)	(1.4160, 3.7259)
8	13	3.250	11.913	5.759	0.1729	(7.8861, 15.0723)	(3.0242, 8.0800)
12	15	7.514	14.502	6.502	0.3685	(9.6937, 20.713)	(3.3711, 11.0059)
16	16	10.720	17.221	7.220	0.5046	(11.262 , 23.9723)	(3.2841, 11.7958)
20	18	12.123	18.586	7.474	0.4658	(12.4273, 26.0259)	(3.5082, 12.5103)
24	21	13.160	18.079	6.650	0.5164	(11.9995, 27.4423)	(2.9936, 12.9422)
28	22	13.211	20.143	7.415	0.5277	(12.9938, 30.4224)	(3.1752, 14.4617)
32	23	17.810	20.668	6.754	0.6432	(14.2238, 30.9350)	(2.9486, 13.4220)
36	24	17.674	22.809	7.809	0.7049	(14.7515, 35.0065)	(3.4593, 16.5423)
40	26	20.433	20.524	5.139	0.7097	(14.6201, 29.5781)	(2.3474, 11.0748)
44	28	19.677	21.368	5.653	0.4725	(15.1086, 31.1136)	(2.4941, 11.7409)
48	29	19.763	22.563	6.011	0.4811	(16.3179, 33.9801)	(2.7222, 13.1064)
52	31	22.563	20.349	3.574	0.3283	(15.4573, 26.4604)	(1.7592, 6.6017)
56	37	20.347	17.660	2.524	0.2461	(13.6996, 22.9458)	(1.3381, 4.1689)
60	41	19.103	16.813	2.178	0.2395	(13.1487, 21.1423)	(1.2499, 3.5165)
64	42	19.564	17.398	2.159	0.2014	(13.5150, 21.6437)	(1.1572, 3.5082)
68	43	19.970	17.976	2.162	0.2245	(14.3584, 21.9799)	(1.1353, 3.6621)
72	43	20.609	19.031	2.286	0.2095	(14.8332, 23.2638)	(1.2229, 4.0922)
76	43	21.468	20.227	2.552	0.2083	(15.9700, 24.4259)	(1.1451, 4.5890)
80	46	20.634	19.939	2.547	0.2594	(15.7134, 24.6610)	(1.2814, 4.9825)
84	46	21.382	21.171	2.910	0.3056	(16.6128, 26.3721)	(1.4181, 6.1385)
88	47	21.864	22.348	3.624	0.3587	(17.6616, 30.0791)	(1.6335, 8.8366)
92	52	20.529	21.118	3.425	0.3639	(16.7124, 28.4620)	(1.4551, 8.8328)
96	56	19.939	20.068	2.925	0.2965	(16.2543, 27.4053)	(1.3364, 8.4232)
100	57	20.232	20.480	2.936	0.2912	(16.5153, 27.8610)	(1.3090, 8.2241)

This section considers the effects of the user input on the estimates made using the Fuzzy-PLE. The influence of incorporating the expert knowledge *ek* is investigated when *ek* is equal to the mean of the distribution, as well as when it is significantly larger and significantly smaller than the mean of the distribution. Also, the effect of changing the confidence in the expert knowledge is considered for the case when *ek* is equal to the mean of the distribution. Finally, the effects of changing the "optimism" parameter *U* is considered. Since the previous estimates were all made with the moderate value of *U* set equal to 0.65, the two ends of the spectrum are considered by setting *U* equal to 0.99 and 0.51. It is seen that in all cases the estimate converges to approximately the same value as enough data becomes available.

6.4 Chapter Summary

In Section 6.2 tests the effectiveness of the Fuzzy-PLE over time are considered. A simulation is developed to model the failure times of several large pieces of equipment over time. Units are put into service, failures are recorded, and at various times estimates are made of the mean survival time from the available data. Several issues are addressed with this simulation. It is demonstrated that for early times the estimate of the mean survival time obtained using the Fuzzy-PLE is superior to the estimate obtained using the PLE and that over time, with matured data, the estimates obtained using the Fuzzy-PLE agree with those made using the PLE.

Section 6.3 considers the effect of users' input, in the form of the three parameters, on the estimates made using the Fuzzy-PLE. In sections 5.2 and 5.3 all of the estimates are made with the parameters *U* set to 0.65, *ek* set to zero and *uc* set to zero. These parameter values indicate that there is a moderate amount of optimism in the continued survival of a censored unit and no expert knowledge is considered in making the estimates. The same data used in the simulation of Section 6.2 are considered with different values of *U*, *ek*, and *uc*. Comparisons are then made, and an understanding of the proper use of these parameters is established. Simulations are run with the expert knowledge, *ek*, input at twice the actual mean and half the actual mean. In both cases the confidence, *uc*, is set very high at 90. Also simulations are run with *ek* and *uc* set at zero but *U* set at both of its extremes of 0.51 and 0.99. It is seen in all cases that as enough data become available, the results converge to the approximately the same estimate regardless of the user input. For example, at time 88 the low of 19.549 is obtained with *ek* equal to 12 and *uc* equal to 90, and the high of 27.141 is obtained with *U* equal to 0.99. In all cases, at time 88 the confidence interval covers 24, the mean of the distribution.

CHAPTER 7

CONCLUSIONS AND FUTURE RESEARCH

7.1 Summary of Research

The research contained in this dissertation is developed to aid reliability engineers in the estimation of survival times when very few data are available and a high proportion of the data is censored. The objective of this research, to develop a computational system that obtains a point estimate of the mean survival time and measures of uncertainty from a very small and highly censored data set has been met. The system uses numerical methods to define fuzzy membership functions about each point that quantify the vague uncertainty due to censoring. An estimate is made using the modified data. Then several methods of quantifying the random uncertainty and the vague uncertainty are developed. The method developed in this research provides an estimate of the survival curve when little data are available and a high proportion of censored values is present in the data set. The result of the method is a survival curve, a point estimate, several measures of the vague uncertainty in the estimate, and a confidence interval that quantifies both random and vague uncertainty. As more complete data become available, the system asymptotically approaches a standard statistical method known as the product-limit estimator. The estimator developed in this work uses heuristic techniques to extract information from the data set and incorporates several forms of expert knowledge into the estimate.

To solve the two problems of obtaining a point estimate and quantifying the uncertainty in the estimate, several theories are incorporated. The estimate of the survival curve with censored data using the Product Limit Estimator (PLE) is used as the basis. The PLE is a maximum likelihood estimator and the method developed in this research converges to the PLE when a complete data set is available. Fuzzy set theory is used to quantify the vague uncertainty in the data that comes from the censoring, by defining a fuzzy membership function about each data point. The data represented by the fuzzy membership functions are then used to make estimates of the survival curve based on the principles used in the PLE. The bootstrap method is modified to quantify the uncertainty, both random and vague. In addition, the methods incorporate heuristic information and expert knowledge. Chapter 2 provides background and a literature review into these and other related theories.

In Chapter 3, the method is developed that estimates the survival curve and then obtains the mean survival time as the area under the curve. For each censored value a fuzzy membership function is used to describe the belief of survivability of the associated unit. These membership functions are generated using characteristics of the data set including the censoring time, size of the data set, proportion censored, the magnitude of other values in the data set, and the total operating time. The membership function for a failed time is a step function equal to one from time zero to the recorded time of failure and zero from the fail time and beyond.

The data set, defined in terms of the membership functions, is used to estimate a survival curve. This estimator is based on the concept of the PLE. The difference is in the data. With the Fuzzy-PLE the data is in the form of membership functions that represent a belief in the survivability of the equipment. If this belief is less than one, then there is some belief of failure. Thus at any point x in the time axis, if any membership function $\mu_i(x)$, $i=1,\ldots, n$ exists such that $0 < \mu_i(x) < 1$, then there is a belief that some failure may exist. As with the PLE the Fuzzy-PLE recalculates the survivability in the presence of the failure. This calculation is done for all points until the survival curve approaches zero (no further belief of survival).

The method also allows for input from the user. The user has the ability to input expert knowledge (*ek*) of the survival time and an associated confidence (*uc*) in the expert knowledge. *uc* is the confidence in the correctness of the expert knowledge measured on a scale of 0 to 100. This information can be given if the user has an idea of the survival time for units under study. The method allows the user to provide another type of input. The optimism parameter, *U*, allows the user to provide input into the method regarding the optimism of continued survival of the censored units. This optimism is input as a value between 0.50 and 1.00 and indicates the amount of optimism the user has in the continued survival of the censored units. Given this information, the membership functions generated from the data are adjusted in accordance with this input.

The Fuzzy-PLE developed in Chapter 3 is used to obtain an estimate of the mean survival time. By definition an estimate has uncertainty associated with it. In Chapter 4 methods are considered to quantify and describe the uncertainty in the estimate of the mean survival time estimated using the Fuzzy-PLE. The uncertainty in the estimate has both random and vague components that represent different phenomena. The random uncertainty is from the randomness of the data, but the vague uncertainty is from the censoring of the units resulting in vague failure times.

Efron's bootstrap methods are used to obtain an empirical distribution of the data and the BC_a method is used to obtain a confidence interval about the mean survival time estimated using the Fuzzy-PLE. Since the Fuzzy-PLE uses fuzzy sets to quantify the uncertainty in the censored values, some measure of the vague uncertainty is needed. Several measures of this vague uncertainty are developed. The difference (S) between the estimate made using the Fuzzy-PLE and that made using the pessimistic view of the data is considered and provides a straightforward measure of the amount of absolute vague uncertainty in the estimate. The relative vague uncertainty S_R provides a measure of the proportion of vague uncertainty used in the estimate. Since S is an estimate of the vague uncertainty another measure is to consider the bootstrap confidence interval about S. Another measure is the mean vague uncertainty $\overline{S}$ of the bootstrap replications. This measure of vague uncertainty estimates the average vague uncertainty in the empirical distribution from the bootstrap replications. Finally, since it is recognized that the amount of vagueness changes across the confidence interval, the two dimensions of uncertainty are considered graphically as an analytical tool.

Simulations are performed in Chapter 5 to test the performance of the Fuzzy PLE. Data is randomly generated from several distributions and estimates are made using the Fuzzy-PLE and associated uncertainty measures. Multiple runs made for each distribution and the data is analyzed. Further, comparisons are made with the actual mean of the distribution and the performance of the PLE on each data set. This chapter tests the performance of the estimator and provides a statistical analysis of the performance of the Fuzzy-PLE relative to the PLE under several conditions.

Tests of the effectiveness of the Fuzzy-PLE over time are also considered. In Chapter 6, a simulation is developed to model the failure times of ten large pieces of equipment over time. Units are put into service, failures are recorded, and at increments of four time units, estimates are made of the mean survival time from the available data. These estimates are made for times four through one hundred. It is demonstrated that for early times the estimate of the mean survival time obtained using the Fuzzy-PLE is superior to the estimate obtained using the PLE and that over time, with matured data, the estimates obtained using the Fuzzy-PLE agree with those made using the PLE. The effect of users' input on the estimates made using the Fuzzy-PLE is also considered. The same data used in the simulation described above are used with different values of U, ek, and uc. Comparisons are then made and an understanding of the proper use of these parameters is established.

The Fuzzy Product-Limit Estimator allows estimates to be made when very few data are available and a high proportion of the data is censored. The estimator is grounded in statistical theory, converging to an established statistically sound estimator when a complete data set is available. The estimator is also flexible when a complete data set is not available. It is seen in this research that the Fuzzy-PLE out performs the PLE under the intended circumstances. Parameters allow the user to incorporate expert knowledge into the estimates in several ways. The user can input information about the failure time of the equipment in question, or a less precise but more intuitive input about the optimism in continued survival of the censored units can be used.

7.2 Future Work and Recommendations

The work performed in this research draws expertise from several diverse areas. The concepts require an understanding of computationally intensive methods in statistics, reliability theory, fuzzy set theory and fuzzy membership functions, and artificial intelligence with its use of heuristics, utility theory, and non-deterministic optimization. Future work in this area can emphasize any one or some combination of these areas.

A genetic algorithm may be developed to decide from several possible membership functions. The concept of using a genetic algorithm to decide from several possible membership functions suggests that several possibilities for membership functions exist, and an evaluation function provides preference to one membership function over another. In the current work only two membership functions are developed. The inverted sigmoid function, which is the main development of this work and the trapezoidal method used mainly as an illustrative point in Chapter 3. The trapezoidal method does provide better results than the inverted sigmoid function and the PLE in a few cases. This improved performance usually occurs when all of the times are censored very early. Thus, the development of other possible membership functions is needed. The random uncertainty is quantified in both the standard error and the confidence intervals obtained using the bootstrap method and vague uncertainty is quantified using several techniques presented in Chapter 4. However, a combination of these resulting in a single value is needed in the development of an evaluation function for the genetic algorithm.

The development of other membership functions can include the concept of a different functional form of the membership function for each censored data. For example, some of the censored data may use the inverted sigmoid function developed in this dissertation and data censored very early relative to the other data may use the trapezoidal membership function. In addition, if it is known that a particular piece of equipment experienced an unusual shock that caused it to fail early, it may be desirable to represent that failure time with another type of fuzzy membership function to model this particular phenomena. Allowing for the concept of individual censored values having different functional forms of membership functions, may necessitate the use of a genetic algorithm. For instance, with just four membership functions to choose from and ten censored data points, there exists 1,048,576 possible combinations. Testing each of these may be computationally prohibitive; thus a genetic algorithm could be used.

The current research is developed and intended to overcome some of the problems encountered when very few data exist and a high proportion of the data is censored. The simulations of Chapter 5 are used to demonstrate that these improvements are accomplished. Then the simulations of Chapter 6 show how the estimates change over time. In these simulations, as more data (information) becomes available the estimate changes, becoming more precise. Essentially, as more information becomes available a better answer is obtained. In this research only reliability data are considered, and the answer is in the mean survival time. A generalization of this concept to other domains represents the most significant possibilities for continued research.

Currently, the question being answered has a binary form of failed or survived. Another domain may include more possibilities. In sample surveys the answer may involve "yes," "no," or "undecided." Thus, by considering other information a fuzzy membership function may be developed about an "undecided" that may determine whether the "undecided" is closer to "yes" or closer to "no." Further, by understanding the domain there may be a way to convince the "undecided" to answer "yes" or "no" in the future. For example, if the undecided in a government office election are known to have sympathy toward an environmental issue (this could be determined through other questions), an advertisement may be enough to sway that vote to a "yes" vote. Then the fuzzy estimator may help to determine if it is cost efficient to use funds to pay for the advertisement.

Another domain to which the current method lends itself is in image analysis. The "gray scale" of a pixel is often determined by adjacent pixels when analyzing images. By considering adjacent pixels, the current pixel may be determined. A fuzzy set is formed about each adjacent pixel representing its gray scale. This information is then aggregated to determine the gray scale of the current pixel. This process is then iterated on all of the pixels until the "best" image is achieved.

The most generalized version of the concept suggested by and used in the Fuzzy Product-Limit Estimator is to consider it as a learning algorithm. As more information becomes available the learner or "student" obtains a better understanding of the subject (gets a better estimate), until one converges to an accurate answer. The technique is to adjust the fuzzy membership functions of several indicators of learning and then evaluate the aggregation of these membership functions until it is determined the subject is understood. The future research in this area requires an understanding of cognitive science and techniques from artificial intelligence. This represents the richest area of continued research since it is the most generalized.

APPENDIX A

EXPONENTIAL 48 DATA SETS

******* 1 *******	
1.037	0.0
2.067	1.0
5.989	0.0
7.370	1.0
7.995	1.0
10.015	0.0
16.630	0.0
20.895	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 2 *******	
3.499	0.0
3.810	0.0
8.844	1.0
11.657	0.0
20.190	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
****** 3 *******	
0.206	1.0
0.765	1.0
2.996	1.0
10.264	0.0
12.971	0.0
21.004	0.0
23.794	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 4 ******	
1.600	0.0
3.090	1.0
3.201	0.0
3.615	0.0
5.821	1.0
6.267	0.0
7.291	1.0
16.709	0.0
17.733	1.0
18.179	0.0
20.385	1.0
20.799	1.0
20.910	0.0
22.400	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 5 ******	
1.127	0.0
2.457	0.0
4.973	0.0
5.771	0.0
6.818	0.0
7.180	1.0
11.048	1.0
12.952	0.0
16.820	0.0
17.182	1.0
18.229	1.0
19.027	1.0
21.543	1.0
22.873	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 6 ******	
2.465	0.0
4.059	1.0
5.083	1.0
5.093	1.0
10.405	1.0
13.595	0.0
18.907	0.0
18.917	0.0
19.941	0.0
21.535	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 7 ******	
1.260	1.0
2.675	0.0
3.347	0.0
4.692	0.0
7.816	0.0
16.184	1.0
17.978	1.0
19.308	1.0
22.740	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 8 ******	
3.362	1.0
8.618	1.0
15.382	0.0
20.638	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 9 ******

0.328	0.0
1.766	0.0
3.704	0.0
8.648	0.0
8.926	1.0
9.262	0.0
9.607	1.0
11.034	1.0
13.308	0.0
14.393	0.0
15.352	1.0
23.672	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 10 ******

0.761	1.0
3.601	0.0
4.119	0.0
5.142	0.0
8.474	0.0
8.733	1.0
10.007	0.0
11.139	1.0
13.993	1.0
14.765	0.0
15.267	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 11 ******

0.309	0.0
0.492	1.0
1.638	1.0
4.402	0.0
4.912	0.0
5.602	0.0
6.463	0.0
7.556	0.0
9.826	0.0
10.166	1.0
11.833	1.0
12.167	0.0
13.834	0.0
14.174	1.0
16.444	1.0
17.537	1.0
17.960	0.0
18.286	0.0
18.398	1.0
24.000	1.0
24.000	1.0

******* 12 ******

1.268	1.0
3.859	1.0
5.363	0.0
8.073	0.0
9.546	1.0
10.109	1.0
13.891	0.0
14.454	0.0
15.927	1.0
18.637	1.0
20.141	0.0
22.732	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 13 ******

0.751	0.0
5.222	0.0
6.367	1.0
8.228	1.0
8.920	0.0
9.108	1.0
15.772	0.0
17.633	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 14 ******

3.807	0.0
3.808	1.0
6.733	0.0
7.312	1.0
8.618	1.0
9.180	1.0
11.887	0.0
12.113	1.0
13.459	0.0
14.820	0.0
15.382	0.0
16.688	0.0
20.193	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 15 ******

6.673	1.0
9.164	0.0
9.490	1.0
14.510	0.0
14.836	1.0
17.327	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 16 ******

2.872	1.0
6.536	0.0
6.650	0.0
8.033	1.0
9.234	1.0
14.478	0.0
14.766	0.0
15.967	0.0
17.464	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 17 ******

1.049	0.0
6.809	0.0
17.191	1.0
22.951	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

*******18 ******

9.974	1.0
10.713	0.0
13.287	1.0
14.026	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 19 ******

1.610	0.0
3.357	0.0
5.718	0.0
6.303	1.0
8.622	0.0
9.225	1.0
11.638	0.0
12.362	1.0
14.775	0.0
22.390	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 20 ******

11.580	0.0
12.420	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 21 ******

1.959	1.0
2.487	0.0
2.770	0.0
4.187	0.0
6.270	0.0
7.694	1.0
9.120	0.0
12.119	0.0
14.880	1.0
14.960	1.0
19.554	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 22 ******

0.638	1.0
3.094	1.0
6.311	0.0
6.967	0.0
7.361	1.0
8.387	0.0
8.868	0.0
12.038	0.0
15.613	1.0
16.639	0.0
17.033	1.0
17.051	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 23 ******	
1.288	0.0
1.477	0.0
2.469	1.0
5.126	1.0
5.290	1.0
17.422	0.0
18.874	0.0
21.531	0.0
22.523	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 24 ******	
0.178	1.0
0.955	1.0
2.997	1.0
3.423	0.0
8.773	0.0
15.227	1.0
19.621	0.0
21.003	0.0
23.822	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 25 ******	
0.508	0.0
4.780	0.0
6.087	0.0
7.032	1.0
8.541	1.0
9.047	0.0
11.865	1.0
12.135	0.0
14.953	1.0
15.459	0.0
16.968	0.0
17.913	1.0
19.220	1.0
23.492	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 26 ******	
0.585	0.0
6.770	1.0
7.228	0.0
8.665	0.0
10.322	1.0
10.501	0.0
10.605	0.0
13.395	1.0
13.499	1.0
13.678	0.0
15.335	1.0
16.645	0.0
16.772	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 27 ******	
2.571	1.0
2.792	1.0
3.731	0.0
5.434	1.0
7.339	0.0
10.745	0.0
13.255	1.0
16.661	1.0
17.478	0.0
18.566	0.0
21.429	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 28 ******	
1.622	1.0
2.735	1.0
9.559	1.0
14.441	0.0
21.265	0.0
22.378	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 29 ******	
0.297	1.0
3.403	1.0
4.285	0.0
4.961	1.0
5.331	0.0
7.526	0.0
8.654	0.0
9.955	0.0
10.643	0.0
14.754	0.0
15.050	0.0
16.474	1.0
18.669	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 30 ******	
0.288	0.0
0.707	1.0
4.295	1.0
8.874	0.0
9.410	1.0
11.467	0.0
12.533	1.0
14.590	0.0
14.838	1.0
19.705	0.0
23.293	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 31 ******

0.902	1.0
1.429	0.0
3.164	0.0
5.218	1.0
7.867	1.0
15.618	0.0
16.133	0.0
22.571	1.0
23.098	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 32 ******

1.661	1.0
3.147	1.0
6.612	1.0
11.009	1.0
12.991	0.0
17.388	0.0
20.853	0.0
22.339	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 33 ******

0.012	0.0
0.554	1.0
1.675	0.0
1.931	0.0
4.204	0.0
5.771	1.0
7.338	0.0
8.293	1.0
9.560	0.0
16.662	1.0
18.229	0.0
21.771	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 34 ******

2.396	0.0
4.973	1.0
6.427	0.0
7.977	1.0
8.206	1.0
15.177	1.0
15.794	0.0
16.023	0.0
19.027	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 35 ******

1.607	0.0
3.479	0.0
20.521	1.0
22.393	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 36 ******

1.471	1.0
4.461	0.0
6.452	1.0
8.656	1.0
13.087	0.0
15.344	0.0
22.529	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 37 ******

1.575	0.0
1.766	0.0
2.045	1.0
3.207	0.0
10.110	0.0
10.807	1.0
11.427	0.0
13.890	1.0
20.793	1.0
21.955	0.0
22.425	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 38 ******

0.277	1.0
8.359	0.0
15.641	1.0
23.723	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 39 ******

2.165	0.0
4.221	0.0
8.205	0.0
8.209	0.0
9.405	1.0
10.474	0.0
11.235	1.0
11.916	0.0
12.084	1.0
12.765	0.0
13.526	1.0
15.795	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 40 ******

0.173	0.0
1.584	1.0
4.702	0.0
7.211	1.0
16.789	0.0
19.298	1.0
22.416	0.0
23.827	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 41 ******

0.162	0.0
2.664	1.0
4.232	0.0
7.177	0.0
7.238	1.0
10.206	1.0
10.548	1.0
13.452	0.0
13.632	0.0
16.762	0.0
16.823	1.0
17.104	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 42 ******

1.258	1.0
5.777	0.0
6.264	1.0
6.354	1.0
11.552	0.0
12.448	1.0
17.646	0.0
17.736	0.0
18.223	1.0
22.742	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 43 ******

0.730	1.0
2.199	1.0
3.763	0.0
7.774	1.0
16.226	0.0
20.237	1.0
21.801	0.0
23.270	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 44 ******

1.450	1.0
1.679	1.0
4.905	1.0
6.989	0.0
17.011	1.0
19.095	0.0
22.321	0.0
22.550	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 45 ******

2.036	1.0
4.027	0.0
4.690	1.0
5.362	0.0
5.571	1.0
5.770	0.0
6.469	1.0
7.213	0.0
7.297	0.0
11.897	0.0
12.103	1.0
14.751	0.0
17.531	0.0
19.310	0.0
19.973	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 46 ******

1.664	0.0
2.725	1.0
3.007	1.0
6.116	1.0
6.155	0.0
10.305	0.0
13.695	1.0
17.845	1.0
17.884	0.0
20.993	0.0
21.275	0.0
22.336	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 47 ******

0.343	0.0
1.721	0.0
7.134	0.0
8.715	0.0
13.565	1.0
16.866	1.0
23.657	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 48 ******

0.171	1.0
3.208	0.0
10.078	1.0
13.922	0.0
20.792	1.0
23.829	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 49 ******

0.386	0.0
6.200	0.0
7.717	0.0
10.133	1.0
13.867	0.0
16.283	1.0
17.800	1.0
23.614	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 50 ******

1.168	0.0
2.568	1.0
4.363	0.0
7.188	0.0
7.423	1.0
8.340	1.0
14.243	0.0
15.409	0.0
15.660	0.0
19.637	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 51 ******

0.889	0.0
1.659	0.0
1.955	0.0
4.972	0.0
5.157	1.0
7.219	0.0
9.046	0.0
13.295	1.0
16.781	1.0
17.073	1.0
18.843	0.0
23.111	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 52 ******

0.387	0.0
2.326	0.0
2.605	1.0
4.768	0.0
5.453	0.0
7.595	1.0
13.228	0.0
16.405	0.0
19.232	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 53 ******

0.707	1.0
1.374	0.0
2.687	0.0
5.052	1.0
9.409	1.0
14.591	0.0
18.948	0.0
21.313	1.0
22.626	1.0
23.293	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 54 ******

3.359	1.0
5.898	1.0
7.591	0.0
10.612	1.0
13.388	0.0
16.409	1.0
18.102	0.0
20.641	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 55 ******

0.808	0.0
0.874	1.0
1.014	0.0
1.927	1.0
3.306	0.0
4.803	1.0
5.397	0.0
8.749	0.0
8.980	0.0
9.071	0.0
9.935	0.0
14.065	1.0
14.929	1.0
18.871	1.0
19.197	0.0
22.073	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 56 ******

0.851	0.0
0.989	0.0
2.622	1.0
2.970	1.0
2.998	1.0
3.386	0.0
5.600	0.0
7.947	1.0
9.446	1.0
14.554	0.0
16.053	0.0
16.626	0.0
17.549	1.0
21.030	0.0
21.378	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 57 ******

1.046	1.0
2.894	0.0
3.610	1.0
7.544	0.0
9.252	1.0
14.748	0.0
15.410	0.0
17.496	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 58 ******

10.111	0.0
13.889	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 59 ******

0.712	0.0
1.431	0.0
2.449	1.0
4.813	0.0
8.951	1.0
10.819	1.0
13.181	0.0
15.049	0.0
19.187	1.0
20.839	0.0
22.569	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 60 ******

2.242	1.0
2.646	1.0
3.836	1.0
4.672	0.0
8.009	0.0
8.087	0.0
8.926	0.0
9.077	0.0
15.074	1.0
15.913	1.0
20.164	0.0
21.354	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 61******

1.074	1.0
6.634	0.0
17.366	1.0
22.926	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 62******

6.127	1.0
7.527	1.0
8.561	1.0
10.965	1.0
10.980	0.0
13.020	1.0
13.035	0.0
15.439	0.0
16.473	0.0
17.873	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 63 ******

2.521	0.0
3.338	0.0
3.701	1.0
5.423	0.0
15.239	1.0
20.299	0.0
21.479	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 64 ******

0.500	0.0
2.130	1.0
2.187	0.0
6.627	1.0
9.510	1.0
14.490	0.0
17.373	0.0
21.813	1.0
21.870	0.0
23.500	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 65 ******	
0.297	0.0
1.365	1.0
2.380	0.0
2.526	0.0
5.094	0.0
6.439	1.0
6.975	1.0
7.261	0.0
8.650	0.0
9.229	1.0
11.605	0.0
11.791	0.0
11.930	0.0
12.209	1.0
14.771	0.0
16.739	1.0
17.561	0.0
21.474	1.0
23.703	1.0
24.000	1.0
24.000	1.0
******* 66 ******	
1.654	1.0
2.309	0.0
3.463	0.0
9.267	1.0
9.659	1.0
14.341	0.0
14.733	0.0
20.537	1.0
21.691	1.0
22.346	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 67 ******	
0.677	1.0
1.737	1.0
4.154	0.0
6.110	0.0
7.868	1.0
8.441	1.0
15.559	0.0
16.132	0.0
17.890	1.0
19.846	1.0
22.263	0.0
23.323	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 68 ******	
0.536	0.0
5.041	1.0
5.858	0.0
6.838	0.0
8.092	1.0
10.415	1.0
13.585	0.0
15.372	0.0
17.162	1.0
18.142	1.0
18.959	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
****** 69 *******	
0.441	0.0
23.559	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 70 ******	
2.102	1.0
3.193	1.0
4.456	1.0
5.116	0.0
10.485	1.0
13.515	0.0
18.884	1.0
19.544	0.0
20.807	0.0
21.898	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 71 ******	
0.054	0.0
0.923	0.0
2.684	0.0
2.977	0.0
4.029	1.0
4.206	0.0
5.552	1.0
5.862	0.0
7.501	0.0
7.970	0.0
10.801	0.0
13.199	1.0
17.215	1.0
19.794	1.0
19.917	0.0
21.316	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 72 ******	
1.485	0.0
2.378	1.0
21.622	0.0
22.515	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 73 ******	
0.047	0.0
8.010	0.0
10.474	0.0
13.526	1.0
15.990	1.0
23.953	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 74 ******	
0.076	1.0
0.648	1.0
0.671	0.0
0.971	0.0
4.987	0.0
5.754	0.0
7.574	0.0
10.286	0.0
12.395	0.0
15.378	0.0
18.246	1.0
19.013	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 75 ******	
0.378	0.0
1.262	0.0
3.103	1.0
4.078	0.0
4.116	0.0
5.295	1.0
7.709	1.0
15.806	1.0
16.291	0.0
18.705	0.0
20.519	0.0
22.738	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 76 ******	
0.774	0.0
5.547	0.0
10.842	0.0
11.631	0.0
12.369	1.0
13.158	1.0
18.453	1.0
23.226	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 77 ******	
2.589	0.0
7.137	1.0
10.387	1.0
10.777	1.0
13.223	0.0
13.613	0.0
16.863	0.0
21.411	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 78 ******	
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 79 ******	
3.804	0.0
4.971	0.0
6.804	0.0
9.333	1.0
14.667	0.0
17.196	1.0
19.029	1.0
20.196	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 80 ******	
0.205	0.0
0.541	0.0
2.252	0.0
6.402	0.0
7.949	1.0
10.395	0.0
11.455	1.0
12.545	0.0
13.605	1.0
16.051	0.0
17.598	1.0
21.748	1.0
23.459	1.0
23.795	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 81 ******

0.710	0.0
2.546	0.0
8.687	0.0
9.840	1.0
12.057	1.0
14.160	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 82 ******

0.290	0.0
0.995	1.0
1.572	0.0
1.890	1.0
4.323	0.0
19.386	1.0
22.110	0.0
22.428	1.0
23.005	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 83 ******

2.012	1.0
2.129	1.0
2.267	1.0
5.258	1.0
7.375	0.0
14.612	0.0
18.742	0.0
21.733	0.0
21.871	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 84 ******

0.685	0.0
0.827	0.0
3.606	1.0
7.907	1.0
16.093	0.0
19.567	0.0
23.315	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 85 ******

0.941	0.0
2.266	1.0
3.165	1.0
19.895	0.0
21.734	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 86 ******

6.436	0.0
7.995	1.0
9.276	0.0
9.568	0.0
11.472	0.0
12.528	1.0
14.724	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 87 ******

1.648	1.0
8.776	0.0
10.900	0.0
11.452	0.0
15.224	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 88 ******

1.733	0.0
2.213	0.0
2.881	0.0
7.547	0.0
8.735	0.0
9.357	1.0
10.142	0.0
13.572	1.0
13.858	1.0
14.643	0.0
15.265	1.0
21.787	1.0
22.267	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 89 ******

0.171	0.0
1.109	1.0
5.675	0.0
7.672	0.0
16.328	1.0
18.325	1.0
22.891	0.0
23.829	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 90 ******

0.181	1.0
2.925	1.0
11.285	1.0
11.846	0.0
12.154	1.0
12.715	0.0
21.075	0.0
23.819	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 91 ******

2.137	0.0
3.183	0.0
3.937	0.0
4.510	0.0
9.070	1.0
14.930	0.0
18.680	1.0
19.490	1.0
20.063	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 92 ******

0.440	0.0
1.099	0.0
1.289	0.0
3.021	0.0
5.827	1.0
7.479	1.0
8.242	0.0
9.534	0.0
9.931	0.0
13.367	1.0
16.521	0.0
20.979	1.0
22.711	1.0
23.560	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 93 ******

1.815	1.0
8.436	0.0
15.564	1.0
22.185	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 94 ******

2.121	1.0
5.099	0.0
7.497	0.0
11.282	0.0
12.718	1.0
16.503	1.0
16.780	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 95 ******

1.192	0.0
1.278	1.0
22.722	0.0
22.808	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 96 ******

6.615	0.0
7.196	1.0
9.101	0.0
9.103	0.0
9.832	1.0
10.189	0.0
14.168	0.0
14.897	1.0
14.899	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 97 ******

1.710	1.0
8.081	0.0
11.583	0.0
12.417	1.0
14.209	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 98 ******

3.179	1.0
11.173	1.0
11.430	0.0
12.570	1.0
12.827	0.0
20.821	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 99 ******

1.626	0.0
3.405	1.0
4.346	0.0
6.156	0.0
7.351	0.0
7.529	1.0
7.786	1.0
12.303	1.0
14.844	0.0
16.214	0.0
17.844	1.0
20.595	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 100 ******

1.178	0.0
2.459	0.0
2.604	0.0
3.183	1.0
3.553	0.0
3.628	0.0
9.065	1.0
10.017	1.0
12.331	0.0
12.805	0.0
17.264	0.0
20.372	1.0
21.541	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 101 ******

2.460	1.0
3.469	0.0
9.387	1.0
14.613	0.0
20.531	1.0
21.540	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 102 ******

2.050	0.0
2.701	1.0
3.350	1.0
3.588	0.0
3.700	0.0
5.490	1.0
14.922	0.0
19.249	0.0
20.300	1.0
20.650	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 103 *******

1.504	0.0
2.409	1.0
2.429	0.0
3.272	0.0
4.310	1.0
10.550	0.0
11.040	0.0
18.186	0.0
20.728	1.0
21.571	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 104 *******

2.799	0.0
4.096	1.0
5.395	1.0
11.157	0.0
12.843	1.0
18.605	0.0
19.904	0.0
21.201	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 105 *******

0.465	0.0
2.307	1.0
6.126	0.0
7.140	0.0
7.578	1.0
8.065	0.0
8.177	0.0
8.439	1.0
9.283	0.0
15.561	0.0
15.823	1.0
15.935	1.0
17.874	1.0
21.693	0.0
23.535	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 106 *******

4.756	0.0
9.883	1.0
11.206	1.0
11.282	1.0
12.718	0.0
12.794	0.0
14.117	0.0
19.244	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 107 *******

1.313	0.0
4.170	0.0
4.970	1.0
5.570	0.0
18.430	1.0
19.030	0.0
19.830	1.0
22.687	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 108 *******

2.097	1.0
2.491	0.0
4.170	0.0
4.798	1.0
5.338	0.0
7.180	1.0
7.203	0.0
9.965	1.0
14.035	0.0
16.820	0.0
21.903	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 109 *******

0.197	0.0
2.147	0.0
2.329	0.0
3.189	0.0
4.225	0.0
5.744	0.0
5.928	0.0
8.158	1.0
10.332	1.0
11.339	0.0
11.420	0.0
12.921	1.0
18.072	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 110 *******

3.264	1.0
4.048	0.0
6.539	0.0
17.461	1.0
19.952	1.0
20.736	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 111 ******

0.137	0.0
4.014	1.0
19.848	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 112 *******

0.169	0.0
2.059	1.0
3.393	0.0
4.412	0.0
8.407	1.0
11.181	0.0
20.607	1.0
21.941	0.0
23.831	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 113 *******

2.731	0.0
4.624	0.0
7.526	0.0
8.784	1.0
11.972	1.0
12.028	0.0
15.216	0.0
16.474	1.0
19.376	1.0
21.269	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 114 *******

4.800	1.0
5.220	0.0
9.035	0.0
9.442	0.0
13.981	0.0
14.558	1.0
14.965	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 115 *******

2.387	1.0
3.011	1.0
4.293	1.0
5.669	0.0
9.878	0.0
11.111	0.0
18.331	1.0
19.707	0.0
21.613	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 116 *******

0.162	1.0
1.546	0.0
7.953	0.0
9.580	1.0
14.420	0.0
16.047	1.0
22.292	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 117 *******

1.560	1.0
4.256	1.0
19.744	0.0
22.440	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 118 *******

5.577	0.0
9.491	1.0
14.509	0.0
18.423	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 119 *******

2.175	0.0
2.780	0.0
3.799	0.0
6.428	1.0
7.954	0.0
8.294	1.0
10.546	0.0
11.279	1.0
11.907	0.0
13.265	1.0
17.572	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 120 *******

1.955	0.0
4.017	1.0
9.686	1.0
11.876	0.0
12.124	1.0
14.314	0.0
18.028	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 121 *******

0.166	1.0
1.776	0.0
6.636	1.0
8.254	0.0
9.569	0.0
12.655	1.0
15.746	1.0
17.364	0.0
23.834	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 122 *******

0.139	0.0
0.873	0.0
4.606	0.0
9.145	1.0
9.821	0.0
10.221	0.0
10.250	0.0
12.906	1.0
14.179	1.0
23.861	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 123 *******

0.014	1.0
2.787	1.0
6.100	1.0
9.856	0.0
14.130	0.0
17.900	0.0
21.213	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 124 *******

0.233	0.0
0.348	0.0
0.951	1.0
2.869	1.0
6.915	1.0
7.068	0.0
9.384	0.0
14.616	1.0
16.932	1.0
17.085	0.0
20.783	0.0
23.049	0.0
23.767	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 125 *******

1.005	1.0
1.940	0.0
4.010	0.0
4.789	0.0
5.121	0.0
10.707	1.0
11.732	0.0
12.149	1.0
12.268	1.0
13.293	0.0
18.985	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 126 *******

0.710	0.0
1.700	0.0
2.733	1.0
3.539	0.0
5.543	1.0
5.754	0.0
6.629	0.0
7.985	1.0
16.015	0.0
16.757	0.0
17.371	1.0
17.537	1.0
17.728	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 127 *******

1.902	0.0
2.499	0.0
4.306	1.0
4.485	0.0
7.764	0.0
7.951	1.0
8.909	0.0
10.812	0.0
12.592	1.0
13.188	1.0
16.049	0.0
16.236	1.0
19.515	1.0
19.694	0.0
22.098	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 128 *******

0.512	1.0
3.126	0.0
5.387	0.0
9.249	0.0
11.626	1.0
18.613	1.0
23.488	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 129 *******

0.358	0.0
4.513	0.0
8.177	0.0
9.009	0.0
10.918	1.0
13.082	0.0
14.991	1.0
15.823	1.0
19.487	1.0
23.642	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 130 *******

3.892	0.0
4.783	1.0
5.028	0.0
8.103	0.0
15.897	1.0
18.972	1.0
19.217	0.0
20.108	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 131 *******

8.144	0.0
9.498	1.0
9.563	1.0
10.227	0.0
13.773	1.0
14.437	0.0
14.502	0.0
15.856	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 132 *******

1.302	1.0
3.269	1.0
7.939	0.0
8.866	1.0
10.767	1.0
13.233	0.0
15.134	0.0
16.061	1.0
20.731	0.0
22.698	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 133 *******

0.561	0.0
0.809	0.0
5.092	1.0
6.379	0.0
9.798	1.0
10.253	0.0
12.376	1.0
14.202	0.0
17.621	1.0
18.908	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 134 *******

0.721	0.0
1.044	0.0
1.069	0.0
6.684	1.0
11.259	0.0
11.532	1.0
12.468	0.0
12.741	1.0
17.316	0.0
22.235	1.0
22.931	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 135 *******

2.479	0.0
4.351	0.0
5.372	0.0
5.796	0.0
8.017	1.0
10.188	0.0
11.056	0.0
12.944	1.0
16.149	1.0
19.649	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 136 *******

0.886	0.0
4.611	1.0
10.501	1.0
13.499	0.0
19.389	0.0
23.114	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 137 *******

2.147	0.0
3.626	0.0
4.180	0.0
4.366	1.0
5.135	1.0
8.096	0.0
15.239	0.0
15.904	1.0
19.634	0.0
19.820	1.0
21.853	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 138 *******

4.887	1.0
6.130	0.0
6.674	0.0
9.448	0.0
14.552	1.0
17.326	1.0
17.870	1.0
19.113	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 139 *******

2.226	1.0
4.567	1.0
4.729	0.0
5.073	0.0
5.815	1.0
6.189	0.0
9.922	0.0
11.852	0.0
12.738	1.0
13.456	0.0
19.433	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 140 *******

5.625	1.0
11.649	1.0
12.351	0.0
18.375	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 141 *******

0.874	0.0
1.150	0.0
1.165	1.0
1.193	0.0
2.044	0.0
7.439	0.0
9.724	1.0
10.352	1.0
10.577	1.0
11.379	0.0
13.648	0.0
14.276	0.0
15.396	0.0
22.807	1.0
22.850	1.0
23.126	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 142 *******

7.709	0.0
10.396	1.0
10.626	1.0
13.374	0.0
13.604	0.0
16.291	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 143 *******

5.718	1.0
8.045	1.0
10.333	0.0
13.667	1.0
15.955	0.0
18.282	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 144 *******

4.621	1.0
8.510	0.0
11.473	1.0
12.527	0.0
15.490	1.0
19.379	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 145 *******

1.263	0.0
2.378	0.0
5.556	1.0
8.042	0.0
8.495	1.0
15.505	0.0
15.958	1.0
17.181	0.0
21.622	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 146 *******

1.679	0.0
2.453	1.0
4.158	0.0
4.319	1.0
5.896	1.0
6.123	0.0
6.373	1.0
8.606	0.0
11.790	0.0
15.394	1.0
17.877	1.0
18.104	0.0
19.681	0.0
21.547	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 147 *******

2.825	0.0
3.054	1.0
5.131	0.0
5.966	0.0
9.608	0.0
14.392	1.0
18.034	1.0
18.869	1.0
20.946	0.0
21.175	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 148 *******

2.170	0.0
3.127	1.0
3.863	1.0
5.444	1.0
7.219	1.0
7.849	0.0
10.568	1.0
13.432	0.0
16.151	1.0
16.781	0.0
18.556	0.0
20.137	0.0
20.873	0.0
21.830	1.0
24.000	1.0
24.000	1.0
24.000	1.0

****** 149 *******

1.897	1.0
3.142	1.0
5.872	0.0
6.290	0.0
6.302	1.0
17.698	0.0
17.710	1.0
18.128	1.0
20.858	0.0
22.103	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 150 ******

3.763	0.0
6.310	1.0
11.445	0.0
11.811	1.0
12.189	0.0
12.555	1.0
17.690	0.0
20.237	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

APPENDIX B

EXPONENTIAL 24 DATA SETS

********* 1 *****

0.429	1.0
4.030	1.0
4.201	0.0
4.651	0.0
4.883	0.0
5.010	0.0
6.300	1.0
6.404	1.0
7.285	0.0
13.395	0.0
16.285	0.0
17.700	0.0
18.990	1.0
19.117	1.0
19.349	1.0
19.970	0.0
24.000	1.0
24.000	1.0
24.000	1.0

********* 2 *****

0.122	0.0
0.558	0.0
0.802	0.0
1.346	0.0
2.809	0.0
2.956	0.0
3.250	0.0
4.001	1.0
4.410	1.0
4.498	0.0
7.320	0.0
8.710	0.0
9.179	0.0
10.289	0.0
11.009	1.0
12.040	1.0
12.991	0.0
14.943	0.0
16.680	1.0
19.042	1.0
21.044	1.0
24.000	1.0
24.000	1.0
24.000	1.0

********* 3 *****

1.769	0.0
1.946	1.0
2.717	1.0
2.987	0.0
3.253	0.0
3.390	0.0
9.043	0.0
10.140	1.0
12.870	0.0
13.860	0.0
14.957	1.0
21.013	1.0
22.054	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

********* 4 *****

0.036	1.0
0.260	1.0
0.494	1.0
0.729	0.0
1.911	1.0
2.148	0.0
2.800	0.0
2.893	0.0
3.046	0.0
3.698	0.0
4.433	0.0
5.495	0.0
6.769	0.0
7.724	0.0
8.916	1.0
9.711	1.0
10.281	0.0
12.192	0.0
13.560	0.0
13.683	0.0
14.128	1.0
14.171	0.0
15.459	1.0
18.391	0.0
19.567	1.0
23.506	0.0
24.000	1.0

********* 5 *****

1.652	1.0
2.389	1.0
3.092	0.0
3.148	0.0
3.432	0.0
3.578	0.0
3.583	0.0
4.061	0.0
5.911	1.0
6.410	0.0
6.600	1.0
7.991	0.0
10.385	0.0
11.679	0.0
12.917	1.0
14.402	0.0
20.422	1.0
22.348	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

********* 6 *****

0.377	0.0
1.275	0.0
1.843	0.0
2.775	1.0
4.347	1.0
6.521	1.0
17.479	0.0
19.573	0.0
19.653	0.0
22.157	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 7 *****

0.458	1.0
0.637	0.0
1.346	1.0
1.862	1.0
2.729	1.0
4.305	0.0
4.600	0.0
4.737	0.0
4.778	0.0
5.028	1.0
7.977	0.0
8.485	0.0
10.787	0.0
15.515	1.0
18.054	0.0
18.972	0.0
19.263	1.0
19.695	1.0
21.271	0.0
22.138	0.0
23.363	1.0
24.000	1.0

******** 8 *****

3.621	1.0
5.162	0.0
8.017	1.0
8.191	1.0
9.368	1.0
10.822	0.0
11.024	1.0
11.053	1.0
12.947	0.0
12.976	0.0
14.632	0.0
15.809	0.0
20.379	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 9 *****

2.365	0.0
2.582	1.0
3.567	1.0
4.196	0.0
5.134	0.0
5.844	1.0
10.870	1.0
13.130	0.0
14.670	1.0
18.156	0.0
20.433	0.0
21.418	0.0
21.635	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 10 *****

1.179	1.0
1.270	0.0
3.445	1.0
4.623	0.0
4.692	0.0
4.817	0.0
5.766	0.0
6.281	0.0
6.662	1.0
6.907	0.0
8.146	0.0
9.113	1.0
9.121	0.0
10.431	0.0
11.037	1.0
14.274	0.0
19.308	1.0
19.377	1.0
22.730	1.0
22.821	0.0
24.000	1.0
24.000	1.0

******** 11 *****

0.794	0.0
0.851	1.0
2.644	0.0
4.161	0.0
5.095	0.0
5.979	0.0
6.713	0.0
6.744	1.0
10.089	1.0
10.458	0.0
13.911	0.0
14.744	1.0
17.256	0.0
21.356	1.0
23.206	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 12 *****

1.848	1.0
3.867	0.0
5.074	1.0
6.004	1.0
9.014	1.0
10.527	0.0
11.118	0.0
13.473	1.0
17.996	0.0
18.926	0.0
22.152	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 13 *****

0.418	0.0
0.837	1.0
0.879	0.0
1.335	1.0
2.165	1.0
6.237	0.0
8.623	0.0
15.377	1.0
16.884	1.0
21.835	0.0
22.665	0.0
22.745	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 14 *****

0.082	0.0
0.674	0.0
1.643	1.0
2.511	0.0
2.557	1.0
2.950	1.0
4.584	0.0
4.778	1.0
7.092	0.0
7.248	1.0
8.166	1.0
8.358	0.0
10.471	1.0
10.998	1.0
12.129	0.0
13.002	0.0
13.529	0.0
15.642	1.0
15.834	0.0
16.752	0.0
18.932	0.0
19.334	1.0
20.376	0.0
22.357	0.0

******** 15 *****

2.593	1.0
2.756	0.0
3.859	0.0
4.838	1.0
6.194	0.0
6.797	0.0
7.645	0.0
7.989	0.0
9.214	1.0
10.632	1.0
11.517	0.0
13.368	0.0
17.548	0.0
17.806	1.0
21.244	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 16 *****

0.959	0.0
2.280	0.0
3.037	1.0
3.505	0.0
4.240	1.0
4.282	1.0
4.886	1.0
5.521	0.0
5.697	0.0
5.699	1.0
8.435	1.0
14.021	0.0
15.565	0.0
16.255	0.0
17.519	1.0
18.301	0.0
19.114	0.0
20.963	0.0
21.720	1.0
24.000	1.0
24.000	1.0

******** 17 *****

0.339	1.0
0.404	1.0
0.871	1.0
1.809	0.0
3.557	1.0
3.960	0.0
5.576	1.0
6.104	0.0
6.965	1.0
8.359	0.0
8.956	0.0
10.661	0.0
10.798	0.0
11.487	0.0
12.798	0.0
13.000	0.0
17.035	0.0
22.191	1.0
23.129	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 18 *****

1.382	1.0
2.388	1.0
4.481	0.0
6.081	1.0
9.455	0.0
11.625	1.0
12.375	0.0
14.545	1.0
17.919	0.0
18.136	0.0
21.612	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 19 *****

0.767	0.0
1.126	0.0
1.377	1.0
3.005	1.0
5.298	0.0
6.049	1.0
8.201	0.0
8.624	0.0
9.027	0.0
11.038	1.0
11.858	0.0
12.142	1.0
12.962	0.0
14.973	1.0
17.325	0.0
20.228	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 20 *****

1.137	0.0
1.933	1.0
2.302	0.0
4.193	0.0
5.028	1.0
15.572	0.0
18.972	0.0
22.863	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 21 *****

0.065	0.0
0.446	0.0
0.925	0.0
1.177	0.0
2.156	0.0
2.264	1.0
3.634	1.0
4.047	1.0
7.493	0.0
9.027	0.0
10.558	1.0
13.442	0.0
14.973	1.0
16.507	1.0
18.582	0.0
20.366	0.0
21.736	0.0
21.778	1.0
22.823	1.0
24.000	1.0
24.000	1.0

******* 22 ******

1.015	0.0
1.150	1.0
1.324	0.0
1.755	0.0
1.774	0.0
2.719	1.0
5.315	1.0
6.342	0.0
6.432	0.0
6.577	1.0
6.999	0.0
7.825	1.0
10.167	0.0
11.030	1.0
11.226	1.0
12.509	1.0
12.970	0.0
15.649	0.0
16.175	0.0
16.930	0.0
17.001	1.0
21.281	0.0
22.850	0.0
22.985	1.0

******** 23 *****

0.013	1.0
1.385	0.0
4.159	1.0
5.538	1.0
5.742	1.0
6.967	1.0
7.768	0.0
16.232	1.0
17.033	0.0
18.258	0.0
18.462	0.0
19.841	0.0
22.602	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 24 *****

0.696	1.0
3.049	0.0
3.790	1.0
3.839	0.0
6.762	1.0
8.107	1.0
9.110	1.0
10.298	0.0
10.996	1.0
13.004	0.0
13.399	0.0
13.702	1.0
14.890	0.0
15.893	0.0
20.210	0.0
20.254	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 25 *****

1.587	0.0
1.646	0.0
3.709	1.0
4.508	0.0
6.403	1.0
7.600	0.0
8.780	0.0
10.245	1.0
15.220	1.0
17.597	0.0
20.291	0.0
22.413	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 26 ******

0.037	1.0
0.279	0.0
0.316	0.0
0.959	1.0
2.212	1.0
2.443	0.0
3.100	0.0
3.931	0.0
7.387	0.0
7.642	0.0
8.779	1.0
10.695	1.0
13.305	0.0
13.513	1.0
15.221	0.0
16.043	1.0
19.110	0.0
21.509	0.0
21.557	1.0
23.963	0.0
24.000	1.0
24.000	1.0

******** 27 *****

1.003	1.0
1.011	0.0
1.839	0.0
3.179	0.0
3.403	1.0
3.469	0.0
6.849	1.0
8.539	1.0
9.636	0.0
10.981	0.0
14.364	1.0
17.151	0.0
19.818	0.0
20.597	0.0
22.161	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 28 *****

0.019	1.0
0.027	0.0
0.079	0.0
1.334	0.0
2.207	1.0
3.739	1.0
4.156	0.0
5.152	1.0
7.220	0.0
7.296	1.0
7.956	0.0
8.849	0.0
11.027	0.0
11.049	1.0
11.413	0.0
11.601	0.0
12.951	0.0
12.973	1.0
15.946	0.0
16.704	0.0
17.636	0.0
22.666	1.0
24.000	1.0
24.000	1.0

******* 29 ******

2.251	0.0
3.387	0.0
3.856	1.0
6.351	0.0
7.568	1.0
7.672	1.0
8.175	1.0
8.867	1.0
9.977	0.0
10.718	0.0
11.746	0.0
13.282	1.0
14.181	0.0
15.825	0.0
20.144	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 30 *****

0.071	0.0
0.628	1.0
3.387	0.0
3.575	0.0
4.859	0.0
5.045	1.0
5.952	1.0
7.475	1.0
8.503	0.0
9.126	0.0
10.048	0.0
10.750	1.0
11.531	0.0
11.666	0.0
11.922	1.0
12.469	1.0
13.250	0.0
13.323	0.0
14.874	1.0
15.497	0.0
18.048	0.0
24.000	1.0
24.000	1.0

******** 31 *****

0.013	0.0
0.354	1.0
0.462	0.0
0.505	0.0
0.561	0.0
0.822	0.0
0.993	0.0
1.489	0.0
2.737	0.0
3.108	1.0
3.369	0.0
3.570	0.0
4.734	1.0
5.933	0.0
5.974	0.0
6.176	1.0
6.515	0.0
7.927	1.0
7.975	0.0
8.049	0.0
9.419	0.0
10.259	1.0
10.853	1.0
11.594	0.0
11.891	0.0
13.741	0.0
14.581	1.0
18.788	0.0
20.070	0.0
23.538	1.0
24.000	1.0

******** 32 *****

0.294	1.0
1.098	0.0
3.238	1.0
3.957	0.0
4.575	1.0
8.258	1.0
11.170	0.0
11.732	1.0
15.742	0.0
19.425	0.0
20.043	1.0
20.762	0.0
23.706	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 33 *****

0.148	0.0
4.465	1.0
7.914	0.0
9.245	1.0
9.791	0.0
14.209	1.0
14.755	0.0
16.086	1.0
19.535	0.0
23.852	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 34 ******

1.698	0.0
2.116	1.0
3.929	0.0
4.005	0.0
4.695	1.0
6.446	1.0
7.690	0.0
11.834	1.0
12.166	0.0
12.381	1.0
17.554	0.0
18.297	1.0
19.305	0.0
21.884	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 35 ******

1.648	1.0
2.045	0.0
2.584	1.0
2.889	0.0
5.675	1.0
7.037	0.0
8.512	1.0
10.554	0.0
11.554	1.0
12.446	0.0
16.963	1.0
18.325	0.0
21.416	0.0
22.352	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 36 *****

1.409	1.0
3.348	0.0
3.551	0.0
3.876	0.0
5.948	1.0
7.720	0.0
8.161	1.0
8.242	1.0
10.332	0.0
15.758	0.0
15.839	0.0
16.572	1.0
19.243	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 37 *****

1.984	1.0
2.714	0.0
4.553	1.0
5.640	1.0
5.931	0.0
7.074	0.0
8.082	0.0
8.748	1.0
9.662	1.0
12.373	0.0
13.934	0.0
14.338	0.0
15.252	0.0
18.069	1.0
18.360	0.0
21.286	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 38 *****

1.322	0.0
3.099	1.0
3.175	1.0
4.278	0.0
4.536	1.0
4.744	0.0
4.757	0.0
5.692	0.0
7.001	0.0
9.547	0.0
16.143	0.0
18.141	0.0
18.308	1.0
19.256	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 39 ******

1.647	0.0
2.839	0.0
4.293	0.0
4.310	1.0
5.537	0.0
7.059	1.0
7.443	1.0
7.957	0.0
8.490	1.0
8.527	1.0
8.702	1.0
8.984	0.0
11.004	0.0
11.314	0.0
15.473	0.0
15.510	0.0
16.557	0.0
22.353	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 40 *****

0.558	0.0
0.780	1.0
1.111	0.0
2.031	0.0
2.074	0.0
2.751	1.0
4.015	0.0
4.203	0.0
5.738	1.0
5.942	0.0
6.704	0.0
7.308	0.0
8.743	0.0
9.208	0.0
9.217	0.0
11.613	0.0
12.387	1.0
13.227	1.0
18.262	0.0
19.797	1.0
22.331	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 41 *****

0.256	0.0
1.279	0.0
1.419	0.0
3.010	1.0
3.210	1.0
3.425	0.0
3.710	0.0
4.637	0.0
7.167	0.0
7.638	0.0
8.308	0.0
16.002	0.0
16.833	1.0
19.107	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 42 ******

0.503	0.0
1.735	1.0
2.476	0.0
2.779	0.0
3.162	0.0
3.550	0.0
4.258	1.0
4.274	1.0
5.055	0.0
5.057	0.0
5.353	1.0
5.687	0.0
6.345	1.0
7.550	1.0
8.616	0.0
10.121	0.0
16.077	0.0
18.313	1.0
18.647	0.0
19.726	0.0
20.450	1.0
22.265	0.0
24.000	1.0
24.000	1.0

******** 43 *****

0.155	0.0
0.250	0.0
0.698	1.0
1.187	0.0
2.579	1.0
3.092	0.0
3.448	1.0
3.546	0.0
4.101	0.0
4.912	0.0
6.548	1.0
8.720	0.0
8.733	0.0
8.763	1.0
9.254	1.0
9.495	0.0
14.746	0.0
15.237	0.0
19.088	1.0
20.552	0.0
23.302	0.0
23.750	1.0
23.845	1.0
24.000	1.0

******** 44 *****

0.901	0.0
1.558	0.0
1.853	1.0
5.844	0.0
6.865	0.0
7.629	1.0
8.956	0.0
10.061	1.0
11.291	1.0
12.381	0.0
14.143	1.0
16.371	0.0
22.147	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 45 ******

0.882	0.0
1.971	0.0
2.496	1.0
2.652	1.0
4.161	0.0
5.814	0.0
8.315	1.0
9.895	1.0
10.073	0.0
10.157	0.0
10.490	0.0
13.843	1.0
13.927	1.0
14.105	0.0
15.685	0.0
19.533	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 46 *****

0.433	1.0
0.495	1.0
2.281	1.0
2.328	0.0
3.076	0.0
3.234	0.0
3.534	0.0
4.532	0.0
4.893	0.0
5.421	1.0
7.060	0.0
7.443	0.0
8.454	0.0
8.604	1.0
9.134	0.0
9.445	0.0
11.078	1.0
15.396	0.0
18.486	0.0
19.107	1.0
23.567	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 47 *****

1.064	0.0
1.229	1.0
1.922	1.0
2.162	0.0
2.473	1.0
2.990	0.0
3.874	0.0
6.129	0.0
6.917	0.0
9.098	0.0
9.617	1.0
9.801	0.0
10.400	1.0
11.394	0.0
11.438	0.0
14.199	1.0
14.610	0.0
14.902	1.0
16.642	0.0
19.062	1.0
22.078	0.0
24.000	1.0
24.000	1.0

******** 48 *****

0.946	0.0
2.109	0.0
2.455	0.0
3.411	1.0
7.030	0.0
10.279	0.0
10.637	1.0
10.985	1.0
12.417	0.0
13.015	0.0
13.558	0.0
13.721	1.0
19.436	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 49 ******	
0.203	1.0
0.886	1.0
4.345	0.0
6.853	1.0
9.429	0.0
10.310	0.0
13.690	1.0
14.368	0.0
17.147	0.0
19.655	1.0
23.114	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 50 ******	
2.874	1.0
3.209	0.0
4.294	0.0
5.121	0.0
6.067	1.0
6.932	1.0
7.083	0.0
7.176	1.0
7.409	0.0
8.770	0.0
11.663	0.0
11.808	0.0
12.192	1.0
12.337	1.0
12.356	0.0
13.859	0.0
16.917	1.0
17.933	0.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 51 ******	
1.541	0.0
3.626	0.0
4.699	0.0
5.269	1.0
5.586	0.0
7.194	0.0
8.675	0.0
10.576	0.0
12.107	1.0
13.145	0.0
13.424	1.0
15.325	1.0
20.374	1.0
22.459	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 52 ******	
3.938	1.0
3.969	0.0
5.984	1.0
7.096	1.0
7.166	0.0
7.421	1.0
8.568	1.0
10.317	0.0
12.896	0.0
12.936	0.0
13.683	1.0
15.432	0.0
16.579	0.0
18.016	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******* 53 ******	
0.240	1.0
1.686	0.0
2.054	1.0
2.393	0.0
6.381	0.0
6.670	0.0
8.053	0.0
11.272	0.0
12.728	1.0
15.706	0.0
15.934	1.0
17.330	1.0
19.553	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
******** 54 *****	
0.294	0.0
1.828	0.0
2.168	0.0
2.686	0.0
3.236	1.0
4.293	0.0
4.961	0.0
5.032	0.0
5.714	0.0
5.769	0.0
8.874	1.0
13.788	0.0
15.126	0.0
18.231	1.0
18.286	1.0
18.968	1.0
19.039	1.0
19.707	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 55 *****

1.568	0.0
1.751	0.0
3.619	1.0
5.635	1.0
5.765	1.0
5.895	1.0
6.322	1.0
7.327	0.0
10.227	0.0
10.293	1.0
11.678	0.0
12.322	1.0
13.354	1.0
13.707	0.0
13.773	1.0
17.678	0.0
18.105	0.0
18.235	0.0
18.365	0.0
20.381	0.0
24.000	1.0

******* 56 ******

0.483	0.0
0.653	1.0
0.836	0.0
0.984	0.0
1.680	0.0
1.909	1.0
2.738	0.0
3.943	0.0
5.311	0.0
5.500	1.0
5.789	0.0
7.352	0.0
8.343	0.0
8.769	0.0
9.329	0.0
10.076	1.0
10.500	1.0
12.435	1.0
13.924	0.0
14.671	1.0
20.412	0.0
23.016	1.0
23.347	0.0
24.000	1.0
24.000	1.0

******* 57 ******

1.734	1.0
4.166	1.0
5.666	0.0
6.137	0.0
9.673	0.0
14.327	1.0
16.128	0.0
18.334	1.0
19.834	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 58 ******

1.976	0.0
3.074	0.0
3.087	1.0
3.429	0.0
3.721	1.0
3.805	1.0
4.276	0.0
6.418	0.0
9.266	0.0
10.929	0.0
11.886	0.0
17.839	0.0
19.724	1.0
20.571	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 59 ******

0.824	1.0
1.462	1.0
1.636	0.0
2.243	1.0
4.082	0.0
4.623	1.0
5.238	1.0
6.341	1.0
6.971	0.0
7.707	1.0
14.680	0.0
15.566	0.0
16.293	0.0
17.659	0.0
19.377	0.0
20.121	0.0
23.176	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 60 ******

0.480	1.0
2.179	0.0
3.050	1.0
5.909	0.0
6.705	0.0
7.237	0.0
7.481	0.0
7.762	1.0
8.675	1.0
16.039	0.0
16.238	0.0
17.295	1.0
20.950	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 61 ******

1.073	0.0
1.986	1.0
2.390	0.0
3.650	0.0
3.769	0.0
5.124	0.0
6.949	0.0
7.330	0.0
8.648	1.0
9.392	1.0
12.837	1.0
14.608	0.0
20.231	1.0
22.014	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 62 ******

0.797	1.0
1.509	1.0
1.723	0.0
2.900	0.0
6.100	1.0
8.010	0.0
10.315	0.0
13.685	1.0
15.990	1.0
17.900	0.0
19.592	0.0
21.480	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 63 *****

0.060	0.0
1.680	0.0
3.022	1.0
3.386	0.0
3.754	0.0
4.205	0.0
9.160	0.0
10.455	0.0
13.545	1.0
14.780	1.0
19.795	1.0
20.246	1.0
20.614	1.0
20.978	0.0
22.320	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 64 ******

0.873	0.0
1.291	0.0
1.442	0.0
1.576	0.0
1.739	0.0
2.160	0.0
2.372	1.0
2.707	0.0
2.956	0.0
3.273	1.0
3.752	0.0
4.424	1.0
6.376	1.0
7.616	0.0
10.691	0.0
12.722	0.0
17.148	0.0
18.000	0.0
21.044	1.0
21.840	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 65 *****

1.658	0.0
7.965	1.0
8.958	1.0
9.038	0.0
9.161	1.0
11.901	1.0
12.099	0.0
13.383	0.0
14.839	0.0
14.962	1.0
16.035	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 66 *****

0.940	0.0
1.255	0.0
1.738	1.0
3.361	0.0
4.080	1.0
8.411	1.0
8.576	1.0
15.424	0.0
15.589	0.0
16.559	0.0
22.262	0.0
22.745	1.0
23.060	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 67 *****

1.661	1.0
4.791	0.0
4.978	0.0
6.575	0.0
9.363	1.0
10.499	1.0
11.478	1.0
11.991	1.0
12.009	0.0
12.522	0.0
12.634	1.0
13.501	0.0
14.637	0.0
19.022	1.0
22.339	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 68 ******

0.466	1.0
1.853	0.0
2.637	0.0
3.002	0.0
3.127	1.0
3.475	0.0
3.655	1.0
5.702	1.0
6.220	0.0
9.095	0.0
9.384	0.0
9.781	0.0
10.564	0.0
11.513	0.0
11.650	0.0
14.823	0.0
14.905	1.0
22.147	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 69 ******

3.214	0.0
4.550	0.0
5.264	0.0
6.097	1.0
6.320	0.0
6.631	1.0
10.895	0.0
13.105	1.0
13.131	1.0
15.522	1.0
17.369	0.0
17.903	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 70 ******

1.931	1.0
5.412	0.0
5.513	1.0
5.943	0.0
8.698	1.0
9.358	0.0
9.791	0.0
10.278	1.0
10.413	0.0
11.656	0.0
13.722	0.0
14.209	1.0
18.487	0.0
18.588	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 71 ******

3.637	1.0
4.476	1.0
5.201	0.0
8.231	1.0
14.323	0.0
15.769	0.0
20.363	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 72 ******

2.097	0.0
2.247	0.0
3.031	0.0
3.281	1.0
3.442	0.0
3.871	0.0
3.901	1.0
4.868	1.0
5.072	0.0
5.302	0.0
5.620	0.0
5.952	0.0
7.699	0.0
11.413	1.0
12.026	1.0
12.587	0.0
18.048	1.0
18.311	1.0
19.132	0.0
20.099	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 73 ******	
1.480	0.0
1.614	0.0
1.921	0.0
2.557	0.0
4.569	0.0
4.581	0.0
4.924	1.0
7.011	1.0
8.462	1.0
8.606	0.0
10.957	0.0
11.217	1.0
12.586	0.0
12.783	0.0
13.780	1.0
14.432	0.0
22.520	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 74 ******	
1.184	0.0
2.020	0.0
2.856	0.0
3.802	0.0
4.554	1.0
4.974	0.0
5.595	1.0
6.931	1.0
7.148	1.0
9.629	0.0
11.958	0.0
12.042	1.0
13.865	0.0
16.852	0.0
19.446	0.0
21.144	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 75 *****	
0.071	0.0
0.217	0.0
0.624	0.0
0.850	1.0
1.746	0.0
2.694	0.0
2.714	0.0
2.969	1.0
3.322	0.0
3.585	0.0
4.321	0.0
4.607	0.0
5.517	0.0
5.771	1.0
7.860	0.0
8.809	1.0
10.367	1.0
10.477	0.0
11.704	0.0
12.296	1.0
13.633	0.0
15.298	0.0
18.759	0.0
20.415	1.0
23.376	1.0
24.000	1.0
24.000	1.0

******** 76 *****	
1.015	0.0
2.163	0.0
2.201	0.0
4.468	0.0
5.468	0.0
7.655	0.0
15.354	1.0
16.345	1.0
19.532	1.0
21.799	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 77 *****	
0.607	0.0
1.698	1.0
1.878	0.0
2.256	1.0
3.286	0.0
4.417	0.0
4.490	0.0
5.104	0.0
6.394	0.0
6.494	1.0
7.582	1.0
9.623	0.0
9.959	1.0
10.780	0.0
11.029	1.0
12.622	0.0
12.971	0.0
13.220	1.0
15.375	0.0
15.811	0.0
17.506	0.0
18.896	1.0
24.000	1.0
24.000	1.0

******** 78 *****	
0.366	1.0
0.677	0.0
0.765	0.0
0.925	0.0
2.238	0.0
3.486	1.0
3.639	0.0
8.161	1.0
9.167	0.0
11.545	0.0
11.690	1.0
14.156	1.0
14.914	0.0
20.361	1.0
20.514	0.0
21.397	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 79 *****

0.255	1.0
0.672	1.0
0.757	0.0
1.179	0.0
1.331	1.0
3.922	0.0
4.134	0.0
5.126	1.0
6.045	0.0
6.728	1.0
6.874	1.0
7.004	0.0
9.003	1.0
10.575	1.0
13.425	0.0
14.740	0.0
14.997	0.0
16.624	0.0
16.996	1.0
17.126	0.0
17.272	0.0
19.823	0.0
22.571	0.0
22.821	1.0

******** 80 *****

0.325	0.0
0.765	0.0
1.568	0.0
1.808	0.0
4.537	1.0
4.572	0.0
8.163	0.0
8.261	0.0
10.843	1.0
11.166	0.0
12.834	1.0
15.837	1.0
19.463	0.0
20.625	1.0
23.235	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 81 ******

0.897	1.0
3.031	0.0
4.172	0.0
4.708	1.0
5.108	0.0
7.067	1.0
7.360	0.0
8.035	1.0
9.065	0.0
10.233	0.0
10.484	0.0
10.763	1.0
11.825	0.0
12.869	0.0
13.516	1.0
13.609	1.0
15.965	0.0
19.292	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 82 *****

0.277	1.0
0.311	0.0
2.070	1.0
4.094	1.0
4.373	0.0
6.320	0.0
6.433	1.0
7.276	0.0
7.440	0.0
9.869	0.0
10.037	0.0
10.127	0.0
11.269	0.0
12.351	1.0
12.731	1.0
15.299	0.0
23.723	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 83 *****

3.649	1.0
4.044	1.0
4.228	1.0
5.013	0.0
5.365	0.0
6.182	0.0
6.186	0.0
8.735	1.0
8.930	0.0
11.026	0.0
14.169	0.0
14.407	0.0
15.265	0.0
17.814	1.0
18.987	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 84 *****

1.633	0.0
2.413	1.0
5.142	0.0
6.538	0.0
7.683	0.0
7.735	1.0
14.683	1.0
16.265	0.0
16.445	0.0
17.462	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 85 *****

0.367	1.0
3.924	0.0
4.337	0.0
6.089	0.0
8.057	1.0
9.015	0.0
10.648	1.0
11.208	0.0
12.792	1.0
15.943	0.0
17.911	1.0
20.076	1.0
23.633	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 86 *****

0.251	0.0
1.264	1.0
1.786	0.0
2.187	0.0
2.573	1.0
2.988	0.0
3.139	0.0
3.865	0.0
4.830	0.0
9.539	0.0
9.898	0.0
10.013	0.0
11.010	0.0
13.987	1.0
14.102	1.0
16.031	1.0
17.146	1.0
21.427	0.0
22.214	1.0
23.749	1.0
24.000	1.0
24.000	1.0

******* 87 ******

0.021	1.0
0.236	1.0
0.999	1.0
1.341	0.0
1.892	1.0
3.859	0.0
6.497	0.0
8.316	0.0
9.514	0.0
9.913	0.0
10.228	1.0
10.379	1.0
13.621	0.0
14.486	1.0
15.163	0.0
15.664	0.0
22.108	0.0
23.764	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 88 *****

1.863	0.0
2.075	0.0
2.093	0.0
3.825	1.0
6.794	1.0
8.771	1.0
9.013	1.0
9.929	1.0
11.626	0.0
12.374	1.0
12.893	0.0
13.154	0.0
14.071	0.0
17.206	0.0
20.175	0.0
22.137	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 89 ******

1.869	0.0
2.654	0.0
3.188	1.0
4.109	1.0
4.753	0.0
6.126	1.0
6.918	1.0
7.491	0.0
8.954	1.0
10.118	0.0
10.813	1.0
11.319	0.0
13.321	0.0
13.882	1.0
15.046	0.0
15.139	0.0
15.220	0.0
17.082	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 90 ******

0.035	0.0
0.878	0.0
2.010	0.0
3.034	0.0
4.764	0.0
10.234	0.0
10.908	1.0
10.946	0.0
11.493	0.0
12.507	1.0
13.054	1.0
13.092	0.0
13.766	1.0
19.236	1.0
20.966	1.0
21.955	1.0
23.122	1.0
24.000	1.0
24.000	1.0

******* 91 ******

0.507	1.0
3.312	1.0
4.470	1.0
4.513	0.0
4.630	0.0
6.802	1.0
7.863	0.0
9.703	0.0
10.985	0.0
14.856	1.0
16.137	1.0
17.198	0.0
19.530	0.0
23.493	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 92 ******

2.845	1.0
4.049	1.0
5.299	1.0
5.333	0.0
18.667	1.0
18.701	0.0
19.951	0.0
21.155	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 93 *****

0.195	1.0
0.271	1.0
0.291	1.0
0.551	0.0
1.184	0.0
1.401	1.0
1.468	0.0
1.719	0.0
2.002	0.0
2.967	1.0
3.019	0.0
5.030	0.0
5.230	0.0
6.586	0.0
6.754	0.0
7.379	1.0
9.950	1.0
10.175	0.0
11.358	0.0
11.590	0.0
13.499	0.0
19.850	0.0
21.727	0.0
23.805	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 94 *****

0.470	1.0
1.248	0.0
1.673	0.0
1.726	1.0
3.265	0.0
4.610	1.0
5.444	0.0
5.618	1.0
10.492	1.0
10.650	1.0
11.106	0.0
12.260	0.0
12.894	1.0
13.350	0.0
16.883	1.0
18.382	0.0
19.390	0.0
20.735	1.0
22.274	0.0
23.530	0.0
24.000	1.0

******* 95 ******

1.486	0.0
1.809	0.0
2.915	1.0
3.789	1.0
6.009	0.0
6.329	1.0
7.096	0.0
7.146	1.0
7.965	0.0
8.535	1.0
9.005	0.0
10.845	0.0
13.989	0.0
14.995	1.0
15.465	0.0
15.862	0.0
16.035	1.0
20.211	0.0
22.514	1.0
24.000	1.0
24.000	1.0

******* 96 ******

0.306	0.0
0.366	0.0
0.631	1.0
0.814	0.0
0.941	1.0
1.621	0.0
1.866	0.0
4.033	1.0
4.357	1.0
5.830	0.0
6.757	0.0
6.788	1.0
8.104	1.0
10.066	0.0
10.186	0.0
11.643	1.0
16.846	0.0
17.243	1.0
19.154	0.0
19.643	0.0
21.748	0.0
23.059	0.0
24.000	1.0
24.000	1.0

******** 97 *****

0.711	0.0
2.013	0.0
2.965	0.0
3.401	0.0
3.996	0.0
4.322	0.0
4.767	1.0
5.188	0.0
6.586	0.0
7.439	0.0
8.367	1.0
8.839	1.0
9.683	0.0
10.445	0.0
17.992	1.0
23.289	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******** 98 *****

1.307	1.0
3.807	0.0
4.728	0.0
5.234	0.0
5.647	1.0
5.816	1.0
6.019	0.0
11.602	0.0
12.398	1.0
14.174	1.0
18.184	0.0
18.353	0.0
18.766	1.0
19.272	1.0
22.693	0.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 99 ******

0.321	1.0
4.608	0.0
7.770	1.0
8.075	1.0
11.317	0.0
11.654	0.0
12.346	1.0
16.230	0.0
23.679	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 100 ******

0.543	0.0
2.074	0.0
2.740	1.0
3.450	1.0
4.599	1.0
4.803	1.0
5.938	1.0
6.909	0.0
9.671	0.0
18.062	0.0
19.401	0.0
20.550	0.0
21.260	0.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0
24.000	1.0

APPENDIX C

EXPONENTIAL 6 DATA SETS

******* 1 ******		11.025	0.0	7.111	0.0
0.143	0.0	11.448	0.0	7.308	0.0
0.374	0.0	14.399	0.0	7.323	0.0
0.390	0.0	16.906	0.0	8.069	0.0
0.415	1.0	******* 2 ******		8.110	0.0
0.743	0.0	0.046	1.0	8.462	0.0
0.972	0.0	0.176	0.0	8.771	0.0
1.125	1.0	0.227	0.0	8.839	1.0
1.339	0.0	0.494	0.0	8.880	0.0
1.415	0.0	0.514	0.0	10.780	0.0
1.462	0.0	0.537	1.0	12.972	0.0
1.508	0.0	0.554	1.0	13.011	0.0
1.575	0.0	0.758	0.0	14.655	0.0
1.803	0.0	0.852	0.0		
2.013	1.0	1.083	0.0		
2.172	0.0	1.088	0.0		
2.510	0.0	1.109	0.0		
2.541	0.0	1.124	0.0		
2.598	1.0	1.206	0.0		
2.933	0.0	1.231	0.0		
2.951	0.0	1.283	0.0		
2.990	0.0	1.327	1.0		
3.004	0.0	1.330	0.0		
3.033	0.0	1.472	0.0		
3.079	1.0	1.493	0.0		
3.228	0.0	1.517	0.0		
3.285	0.0	1.542	0.0		
3.337	0.0	1.642	0.0		
3.346	0.0	1.671	0.0		
3.347	1.0	2.396	0.0		
3.560	0.0	2.465	1.0		
4.119	0.0	2.488	0.0		
4.137	0.0	2.729	1.0		
4.166	0.0	2.954	0.0		
4.375	0.0	2.966	0.0		
4.466	0.0	3.211	0.0		
4.647	0.0	3.420	0.0		
4.657	0.0	3.765	1.0		
4.716	1.0	3.819	0.0		
5.187	1.0	3.856	0.0		
6.105	0.0	4.017	1.0		
6.253	0.0	4.335	0.0		
6.359	0.0	4.337	0.0		
7.092	1.0	4.391	0.0		
8.228	0.0	5.822	0.0		
8.326	0.0	6.623	0.0		
8.941	0.0	6.819	0.0		
9.991	0.0	6.913	1.0		
10.410	0.0	7.034	0.0		
10.858	1.0	7.068	0.0		

******** 3 *****	
0.060	0.0
0.144	0.0
0.240	0.0
0.555	0.0
0.658	0.0
0.781	0.0
0.948	0.0
0.991	0.0
1.047	0.0
1.159	0.0
1.271	0.0
1.305	0.0
1.796	1.0
1.920	0.0
2.480	0.0
2.609	0.0
3.085	0.0
3.192	1.0
3.469	0.0
3.920	0.0
4.025	1.0
4.325	0.0
4.435	0.0
4.465	1.0
4.743	0.0
4.942	0.0
4.986	0.0
5.151	0.0
5.278	0.0
5.560	0.0
5.671	0.0
5.783	0.0
6.283	1.0
6.312	0.0
6.937	0.0
7.142	0.0
7.374	0.0
7.491	1.0
7.650	0.0
8.205	0.0
8.772	0.0
8.893	0.0
9.854	1.0
10.021	0.0
11.061	1.0
12.676	1.0
13.478	0.0
16.858	1.0

******* 4 ******	
0.046	1.0
0.124	0.0
0.506	0.0
0.612	0.0
0.998	0.0
1.041	0.0
1.058	0.0
1.064	0.0
1.205	0.0
1.239	0.0
1.729	0.0
1.730	0.0
1.782	0.0
1.830	1.0
1.890	0.0
2.018	0.0
2.048	1.0
2.847	1.0
3.002	0.0
3.102	1.0
3.210	0.0
3.331	0.0
3.375	1.0
3.408	0.0
3.578	0.0
3.866	0.0
4.170	0.0
4.830	0.0
4.917	0.0
5.100	0.0
5.190	0.0
5.367	0.0
5.448	0.0
5.529	1.0
5.876	0.0
6.002	1.0
6.097	0.0
6.168	0.0
6.259	0.0
6.338	0.0
6.550	0.0
6.585	0.0
6.744	0.0
7.107	0.0
7.211	0.0
7.464	0.0
7.877	1.0
7.924	0.0
7.965	0.0
8.238	0.0
8.298	0.0
9.664	0.0
10.056	1.0
10.383	0.0

********* 5 ****	
0.203	0.0
0.241	0.0
0.260	0.0
0.737	0.0
1.508	0.0
2.133	0.0
2.696	0.0
3.337	0.0
3.535	0.0
3.565	0.0
3.892	0.0
4.350	1.0
4.458	1.0
4.798	0.0
5.124	0.0
5.196	0.0
5.274	0.0
5.430	0.0
5.455	0.0
6.013	0.0
6.592	1.0
6.972	1.0
7.224	1.0
9.110	0.0
10.995	1.0
11.675	1.0
12.084	0.0
12.702	0.0
12.802	0.0
14.712	0.0
14.890	1.0
16.542	1.0
16.769	0.0
18.726	1.0
********* 6 *******	
0.056	0.0
0.062	0.0
0.240	0.0
0.279	0.0
0.293	0.0
0.507	0.0
0.574	1.0
0.698	0.0
0.811	0.0
1.135	0.0
1.143	0.0
1.331	0.0
1.335	0.0
1.348	0.0
1.542	0.0
1.546	0.0
1.576	1.0
1.666	0.0
1.768	1.0
1.868	0.0
1.874	0.0
2.054	0.0
2.222	0.0
2.348	1.0
2.500	1.0
2.522	0.0
2.913	0.0
2.937	1.0
2.957	0.0
3.084	0.0
3.088	0.0
3.220	0.0
3.233	0.0
3.269	0.0
3.811	0.0
3.966	1.0
4.118	0.0
4.188	0.0
5.548	0.0
6.354	1.0
6.457	1.0
6.470	0.0
6.880	1.0
7.434	0.0
7.844	0.0
7.906	0.0
7.914	0.0
8.092	0.0
8.712	0.0
9.811	0.0
10.075	0.0
10.772	0.0
12.579	0.0
13.652	0.0
14.311	0.0
15.104	0.0
******* 7 ******	
0.029	0.0
0.053	0.0
0.140	1.0
0.231	0.0
0.264	1.0
0.283	0.0
0.351	0.0
0.385	0.0
0.411	0.0
0.477	0.0
0.509	0.0
1.009	0.0
1.526	0.0
1.533	0.0
1.578	0.0
1.679	0.0
2.040	1.0
2.058	0.0
2.325	0.0
2.482	1.0
2.920	0.0
3.836	1.0
4.083	0.0
4.180	1.0
4.198	0.0
4.595	0.0
4.678	1.0
4.734	0.0
4.790	0.0
4.937	0.0
5.580	0.0
6.265	1.0
6.436	0.0
6.613	0.0
6.798	0.0
6.999	0.0
7.141	0.0
7.304	0.0
7.513	0.0
7.789	0.0
8.030	0.0
8.509	0.0
9.271	1.0
9.714	0.0
10.487	0.0
12.001	0.0
12.323	0.0
14.914	0.0
24.000	1.0

******** 8 *****	
0.047	0.0
0.082	0.0
0.202	0.0
0.314	0.0
0.741	0.0
0.796	0.0
0.803	1.0
0.927	0.0
1.002	0.0
1.247	1.0
1.796	0.0
1.953	1.0
1.977	1.0
2.081	1.0
2.170	0.0
2.268	0.0
2.284	0.0
2.461	0.0
2.520	1.0
2.808	0.0
3.413	0.0
3.421	0.0
3.606	0.0
3.905	0.0
4.031	0.0
4.291	0.0
4.531	0.0
4.618	0.0
4.682	0.0
4.914	0.0
5.126	0.0
5.368	0.0
5.604	0.0
6.230	0.0
6.372	0.0
6.446	0.0
6.561	1.0
7.972	0.0
8.527	0.0
8.602	0.0
9.042	0.0
9.073	1.0
9.839	0.0
9.901	0.0
10.738	0.0
14.468	1.0
16.237	0.0
24.000	1.0

******** 9 *****	
0.064	0.0
0.086	0.0
0.124	0.0
0.220	0.0
0.273	1.0
0.709	0.0
0.780	0.0
0.903	0.0
1.077	0.0
1.597	1.0
1.661	0.0
2.464	0.0
2.480	0.0
2.578	0.0
3.377	0.0
3.451	1.0
3.461	0.0
3.804	0.0
3.907	1.0
4.047	0.0
4.138	0.0
4.393	0.0
4.395	0.0
4.415	0.0
4.560	1.0
4.642	0.0
5.760	0.0
5.889	0.0
5.929	1.0
5.976	0.0
6.170	0.0
6.488	1.0
7.106	0.0
7.504	0.0
7.695	0.0
8.722	1.0
9.022	0.0
10.341	0.0
10.396	0.0
10.682	1.0
11.016	1.0
11.640	0.0
13.379	0.0
15.278	0.0
17.404	0.0

******* 10 ******	
0.029	0.0
0.069	0.0
0.526	0.0
0.538	0.0
0.545	0.0
0.567	1.0
1.360	0.0
1.617	0.0
1.778	0.0
2.018	0.0
2.035	0.0
2.193	1.0
2.381	0.0
2.658	0.0
2.793	0.0
3.003	0.0
3.187	1.0
3.283	0.0
3.359	0.0
3.459	0.0
3.540	0.0
3.683	1.0
3.711	0.0
3.742	0.0
4.028	0.0
4.224	0.0
4.234	0.0
5.100	0.0
5.131	0.0
5.785	0.0
5.881	0.0
6.439	1.0
7.333	0.0
7.338	1.0
7.426	0.0
7.504	0.0
7.547	0.0
7.729	0.0
8.347	0.0
8.661	0.0
9.194	1.0
12.303	0.0
12.771	0.0
13.409	1.0
15.158	1.0
22.383	1.0

******* 11 ******	
0.060	1.0
0.072	0.0
0.152	0.0
0.152	0.0
0.166	0.0
0.287	0.0
0.295	0.0
0.308	0.0
0.376	0.0
0.506	0.0
0.616	1.0
0.625	0.0
0.648	0.0
0.695	0.0
1.116	0.0
1.254	0.0
1.279	1.0
1.420	0.0
1.709	0.0
1.738	0.0
1.936	0.0
1.960	0.0
2.305	0.0
2.471	0.0
2.667	0.0
2.668	1.0
2.692	0.0
2.756	1.0
3.043	0.0
3.103	0.0
3.112	0.0
3.499	0.0
3.564	0.0
3.705	1.0
3.717	0.0
3.746	0.0
3.753	0.0
3.802	0.0
4.392	0.0
4.574	0.0
4.649	0.0
4.707	1.0
4.781	0.0
4.853	0.0
5.197	1.0
5.433	0.0
5.560	0.0
5.749	0.0
6.516	0.0
6.554	0.0
6.584	0.0
6.707	1.0
6.943	0.0
7.053	0.0
7.394	0.0
7.419	0.0
8.765	0.0
8.852	0.0
9.674	0.0
9.874	0.0
12.740	0.0
17.057	1.0

****** 12 *******	
0.156	0.0
0.160	0.0
0.351	0.0
0.357	0.0
0.391	0.0
0.526	0.0
0.737	0.0
0.868	0.0
0.890	1.0
1.535	0.0
1.691	1.0
1.695	0.0
1.760	0.0
1.812	0.0
1.837	0.0
2.045	0.0
2.180	0.0
2.429	0.0
2.779	0.0
2.819	1.0
2.966	0.0
3.023	0.0
3.229	0.0
3.306	0.0
3.318	1.0
3.349	0.0
4.051	0.0
4.098	0.0
4.846	1.0
5.931	0.0
6.419	0.0
8.031	0.0
8.599	0.0
8.880	0.0
9.057	0.0
9.507	0.0
9.960	1.0
11.228	1.0
11.327	0.0
11.404	0.0
12.895	1.0
13.252	0.0
14.968	1.0
16.208	0.0
23.132	1.0

******* 13 ******		******** 14 *****		******* 15 ******	
0.472	0.0	0.007	0.0	0.280	0.0
0.582	0.0	0.053	0.0	0.395	0.0
1.131	0.0	0.119	0.0	0.650	0.0
1.177	0.0	0.120	1.0	0.805	0.0
1.268	0.0	0.246	1.0	0.861	0.0
1.289	1.0	0.340	0.0	1.224	0.0
1.608	1.0	0.459	0.0	1.402	0.0
1.618	0.0	0.742	0.0	1.521	0.0
1.722	0.0	0.805	0.0	1.748	0.0
1.827	1.0	0.940	0.0	1.763	0.0
1.921	0.0	1.460	0.0	2.110	1.0
1.951	0.0	1.581	1.0	2.212	0.0
2.014	0.0	1.647	0.0	2.307	0.0
2.046	0.0	1.647	0.0	2.425	0.0
2.080	0.0	1.750	0.0	2.468	0.0
2.095	0.0	2.090	0.0	2.553	0.0
2.211	0.0	2.637	0.0	2.567	0.0
2.696	1.0	2.675	0.0	2.740	0.0
3.038	0.0	2.676	0.0	2.840	1.0
3.111	0.0	2.992	1.0	3.189	1.0
3.480	1.0	3.067	0.0	3.706	0.0
3.534	0.0	3.274	0.0	4.395	0.0
3.567	0.0	3.612	0.0	5.077	0.0
3.620	0.0	3.852	0.0	5.169	1.0
4.058	0.0	3.955	0.0	5.689	0.0
4.084	0.0	4.555	1.0	5.737	0.0
4.281	0.0	4.736	0.0	5.982	0.0
4.354	0.0	5.168	0.0	6.017	0.0
4.378	0.0	5.483	0.0	6.167	0.0
4.659	0.0	5.630	0.0	6.188	0.0
4.813	0.0	5.801	1.0	6.405	0.0
4.925	0.0	5.885	1.0	6.575	0.0
5.076	0.0	5.890	0.0	6.716	0.0
5.169	0.0	6.590	1.0	6.828	1.0
5.285	0.0	6.603	0.0	6.880	0.0
5.932	0.0	7.967	0.0	7.330	0.0
5.950	0.0	8.061	0.0	7.358	0.0
6.832	0.0	8.365	1.0	7.501	0.0
7.522	1.0	8.506	0.0	8.270	0.0
7.694	0.0	8.536	0.0	8.811	0.0
8.097	0.0	9.008	0.0	9.246	0.0
8.461	1.0	10.301	0.0	10.932	1.0
8.550	0.0	10.888	0.0	11.016	1.0
13.672	1.0	11.144	0.0	14.203	1.0
15.539	0.0	12.597	0.0	15.240	1.0
16.239	0.0	12.925	1.0	16.499	1.0
16.306	1.0	13.572	0.0		
18.068	1.0	19.043	0.0		

******* 16 ******	
0.121	0.0
0.152	0.0
0.174	0.0
0.223	0.0
0.273	0.0
0.291	0.0
0.408	0.0
0.476	0.0
0.492	0.0
0.499	0.0
0.533	0.0
0.685	0.0
1.074	0.0
1.195	0.0
1.326	0.0
1.806	0.0
1.810	1.0
2.009	0.0
2.203	1.0
2.280	0.0
2.525	0.0
2.547	0.0
2.802	0.0
2.955	1.0
3.104	1.0
3.380	0.0
3.858	1.0
3.918	0.0
4.034	0.0
4.096	0.0
4.159	1.0
4.406	0.0
4.616	1.0
4.670	0.0
4.738	0.0
4.799	0.0
4.872	0.0
6.419	0.0
6.877	0.0
6.917	0.0
7.001	0.0
7.237	0.0
9.054	0.0
9.161	1.0
9.231	0.0
10.844	0.0
11.407	0.0
13.156	1.0
15.796	0.0
19.387	0.0
24.000	1.0
******** 17 *****	
0.020	1.0
0.044	0.0
0.407	0.0
0.433	0.0
0.500	0.0
0.618	0.0
0.723	0.0
0.739	1.0
0.915	0.0
1.167	0.0
1.483	0.0
1.531	0.0
1.630	0.0
1.665	0.0
1.670	0.0
1.839	1.0
2.095	0.0
2.295	1.0
2.326	0.0
2.510	1.0
2.649	0.0
2.679	0.0
2.734	0.0
2.844	1.0
3.219	1.0
3.356	0.0
3.569	0.0
3.937	0.0
3.945	0.0
4.407	0.0
4.508	0.0
4.993	0.0
5.476	0.0
5.526	0.0
5.561	0.0
5.990	0.0
6.038	0.0
6.877	0.0
6.883	0.0
6.919	0.0
7.124	0.0
7.291	0.0
7.313	0.0
7.372	0.0
7.551	0.0
8.701	0.0
8.717	1.0
9.451	1.0
11.624	0.0
12.696	0.0
13.830	0.0
21.613	1.0

******** 18 *****		5.276	1.0	7.840	0.0
0.076	0.0	5.596	0.0	8.163	1.0
0.127	0.0	5.625	1.0	9.446	0.0
0.169	0.0	5.753	0.0	11.513	0.0
0.220	1.0	5.934	0.0	13.399	0.0
0.359	0.0	6.899	0.0	13.911	0.0
0.360	0.0	7.521	0.0	14.736	0.0
0.445	0.0	7.538	0.0	15.010	0.0
0.479	0.0	7.619	0.0	15.516	0.0
0.560	1.0	7.775	0.0	17.784	0.0
0.588	0.0	9.577	0.0	24.000	1.0
0.654	0.0	9.720	0.0		
0.686	0.0	11.456	0.0		
0.720	0.0	13.669	0.0		
0.945	0.0	16.462	1.0		
0.968	1.0	19.739	0.0		
0.974	1.0	*******19 ******			
1.120	0.0	0.154	0.0		
1.272	0.0	0.340	0.0		
1.391	0.0	0.375	0.0		
1.427	0.0	0.583	0.0		
1.444	0.0	0.821	0.0		
1.458	0.0	1.004	0.0		
1.525	0.0	1.069	1.0		
1.595	1.0	1.135	1.0		
1.739	0.0	1.382	0.0		
1.816	0.0	1.531	0.0		
1.821	0.0	1.762	0.0		
1.900	0.0	1.827	1.0		
1.994	1.0	1.860	0.0		
2.051	0.0	1.868	1.0		
2.113	0.0	1.896	0.0		
2.288	0.0	1.926	0.0		
2.306	0.0	2.310	0.0		
2.510	0.0	2.452	0.0		
2.542	0.0	2.627	0.0		
2.556	0.0	2.644	0.0		
2.848	0.0	2.910	0.0		
3.003	0.0	2.924	0.0		
3.189	0.0	3.139	0.0		
3.415	0.0	3.197	0.0		
3.601	1.0	3.356	0.0		
3.701	0.0	4.663	1.0		
4.388	0.0	5.292	0.0		
4.614	0.0	5.330	1.0		
4.815	0.0	6.455	1.0		
4.867	0.0	7.100	0.0		
5.024	0.0	7.247	0.0		
5.180	0.0	7.502	1.0		

****** 20 *******

0.217	0.0
0.280	0.0
0.373	0.0
0.447	0.0
0.522	0.0
0.538	0.0
0.778	0.0
0.903	1.0
0.930	0.0
0.998	0.0
1.042	0.0
1.052	0.0
1.209	1.0
1.344	0.0
1.348	0.0
1.435	0.0
1.468	1.0
1.600	0.0
2.127	0.0
2.142	0.0
2.258	0.0
2.490	0.0
2.645	0.0
2.845	0.0
2.869	0.0
2.936	0.0
3.078	0.0
3.206	0.0
3.454	0.0
3.583	0.0
3.634	0.0
3.703	1.0
3.852	0.0
4.116	0.0
4.162	1.0
4.430	1.0
4.474	0.0
4.518	0.0
4.685	0.0
5.065	0.0
5.065	0.0
5.186	0.0
5.547	1.0
5.623	0.0
5.971	0.0
6.181	0.0
7.506	1.0
7.788	0.0
7.839	0.0
11.454	0.0
12.108	1.0
12.964	1.0
14.806	0.0
17.453	0.0
21.754	0.0

******* 21 ******

0.119	0.0
0.133	0.0
0.453	1.0
0.479	1.0
0.508	0.0
0.659	0.0
1.021	0.0
1.149	0.0
1.216	1.0
1.583	0.0
1.893	0.0
1.960	0.0
2.046	0.0
2.148	0.0
2.177	0.0
2.186	0.0
2.344	0.0
2.418	1.0
2.465	0.0
2.571	0.0
2.669	0.0
2.759	0.0
2.854	1.0
2.884	0.0
3.355	0.0
3.395	1.0
3.534	0.0
3.604	0.0
3.718	0.0
3.737	0.0
3.784	0.0
3.792	0.0
3.871	0.0
4.080	0.0
4.256	0.0
4.309	1.0
4.619	0.0
5.130	0.0
5.278	1.0
5.813	0.0
6.154	0.0
6.358	0.0
6.549	0.0
6.633	0.0
7.179	1.0
7.420	0.0
7.861	0.0
8.117	0.0
8.157	0.0
8.161	0.0
8.226	0.0
8.841	0.0
9.574	1.0
10.446	0.0
11.162	0.0
12.194	0.0

****** 22 *******

0.249	0.0
0.322	0.0
0.478	1.0
0.643	0.0
0.763	0.0
0.948	0.0
1.229	0.0
1.548	1.0
1.663	0.0
1.914	0.0
1.933	1.0
2.457	1.0
2.476	0.0
2.632	0.0
2.749	0.0
3.174	1.0
3.736	0.0
4.216	0.0
4.287	0.0
4.520	0.0
4.615	0.0
4.616	0.0
4.620	0.0
4.859	0.0
4.988	0.0
5.012	0.0
5.045	1.0
5.193	0.0
5.298	0.0
5.660	0.0
5.893	1.0
6.659	0.0
7.064	0.0
7.344	0.0
7.472	0.0
7.832	0.0
7.841	0.0
7.849	0.0
8.811	1.0
9.480	0.0
10.508	1.0
10.908	0.0
11.949	0.0
15.189	0.0
23.357	1.0

******* 23 ******

0.027	0.0
0.149	0.0
0.192	0.0
0.225	1.0
0.316	0.0
0.616	0.0
0.651	0.0
0.698	0.0
0.709	0.0
0.793	0.0
0.857	0.0
1.023	0.0
1.025	0.0
1.208	0.0
1.412	0.0
1.646	0.0
1.650	1.0
1.753	0.0
1.834	0.0
1.873	0.0
1.994	1.0
2.057	0.0
2.171	0.0
2.568	0.0
2.668	0.0
2.685	0.0
2.712	1.0
3.328	1.0
3.342	0.0
3.389	0.0
3.684	0.0
3.686	0.0
4.036	0.0
4.098	0.0
4.608	1.0
4.871	0.0
4.905	0.0
5.125	0.0
5.137	0.0
5.200	0.0
5.268	0.0
5.328	0.0
6.144	1.0
6.624	0.0
7.930	0.0
8.906	0.0
8.961	0.0
9.302	0.0
9.581	0.0
9.616	0.0
9.730	0.0
9.969	1.0
11.242	1.0
12.007	1.0
12.176	0.0
16.294	0.0

****** 24 *******		5.314	0.0	4.132	0.0
0.080	0.0	5.351	0.0	4.681	0.0
0.090	0.0	5.447	0.0	5.142	0.0
0.216	0.0	6.094	0.0	5.150	0.0
0.226	0.0	6.759	0.0	5.170	0.0
0.387	0.0	7.791	0.0	5.230	0.0
0.490	0.0	7.900	0.0	5.373	0.0
0.522	0.0	7.991	1.0	5.525	0.0
0.665	0.0	8.292	1.0	5.724	0.0
0.694	0.0	8.845	0.0	6.498	0.0
0.743	0.0	9.593	0.0	6.998	0.0
0.837	0.0	9.901	0.0	7.035	0.0
1.054	0.0	10.114	0.0	7.638	0.0
1.086	0.0	13.072	1.0	7.708	0.0
1.287	0.0	13.472	0.0	7.809	1.0
1.319	0.0	******* 25 ******		8.309	1.0
1.397	0.0	0.030	0.0	9.966	1.0
1.460	0.0	0.213	1.0	12.118	0.0
1.463	0.0	0.254	1.0	14.802	0.0
1.578	0.0	0.277	0.0	14.857	1.0
1.664	0.0	0.283	0.0	15.380	1.0
1.751	0.0	0.434	0.0	21.943	1.0
1.766	1.0	0.443	0.0		
2.020	0.0	0.456	0.0		
2.325	0.0	0.498	0.0		
2.342	0.0	0.690	1.0		
2.415	0.0	0.849	0.0		
2.524	1.0	0.880	0.0		
2.526	0.0	0.903	0.0		
2.606	1.0	0.912	0.0		
2.620	0.0	0.915	0.0		
3.009	0.0	1.143	0.0		
3.204	0.0	1.157	0.0		
3.467	0.0	1.398	0.0		
3.544	0.0	1.469	1.0		
3.744	0.0	1.654	0.0		
3.770	1.0	1.811	0.0		
3.784	0.0	1.834	0.0		
3.810	0.0	2.057	0.0		
4.012	0.0	2.371	0.0		
4.124	0.0	2.422	0.0		
4.288	1.0	2.555	0.0		
4.457	0.0	2.726	0.0		
4.468	0.0	2.915	0.0		
4.575	0.0	3.369	0.0		
4.588	1.0	3.837	0.0		
4.871	0.0	3.900	0.0		
4.994	0.0	4.024	0.0		
5.196	1.0	4.131	0.0		

******* 26 ******	
0.063	0.0
0.102	0.0
0.152	0.0
0.190	0.0
0.469	0.0
0.493	1.0
0.972	0.0
1.013	0.0
1.085	0.0
1.208	0.0
1.492	1.0
1.518	0.0
1.665	0.0
1.682	0.0
1.812	0.0
1.868	1.0
2.039	0.0
2.421	0.0
2.719	0.0
2.836	1.0
2.988	0.0
3.361	0.0
3.379	0.0
3.487	1.0
3.611	0.0
3.716	0.0
3.971	0.0
4.083	0.0
4.291	1.0
4.310	0.0
4.340	0.0
4.356	0.0
4.615	0.0
4.810	0.0
5.074	1.0
5.179	0.0
5.236	0.0
6.169	0.0
6.340	0.0
6.511	0.0
6.988	0.0
7.358	1.0
9.148	0.0
9.266	0.0
9.601	0.0
9.716	1.0
9.857	0.0
11.943	0.0
12.127	0.0
12.427	0.0
12.935	1.0
13.008	0.0
******* 27 ******	
0.022	0.0
0.039	0.0
0.101	0.0
0.434	0.0
0.463	0.0
0.542	1.0
0.715	0.0
0.751	0.0
0.783	0.0
1.139	0.0
1.294	1.0
1.371	1.0
1.430	0.0
1.501	0.0
1.604	0.0
1.682	0.0
1.854	0.0
1.897	0.0
2.188	0.0
2.596	0.0
2.664	0.0
2.803	0.0
2.941	0.0
3.325	0.0
3.493	0.0
3.555	0.0
3.560	0.0
3.673	0.0
3.779	0.0
3.824	0.0
4.074	0.0
4.138	0.0
4.481	0.0
4.848	0.0
4.972	1.0
5.181	0.0
5.269	0.0
5.344	0.0
5.588	0.0
5.912	0.0
5.913	0.0
6.992	1.0
7.317	0.0
7.586	0.0
7.812	0.0
8.459	0.0
8.468	0.0
8.804	1.0
9.947	1.0
13.641	1.0
15.597	1.0
16.617	0.0
17.016	1.0

******* 28 ******	
0.105	0.0
0.146	0.0
0.164	0.0
0.237	0.0
0.295	0.0
0.318	1.0
0.351	0.0
0.583	0.0
0.603	0.0
0.822	0.0
0.870	0.0
0.977	0.0
1.214	0.0
1.230	0.0
1.373	0.0
1.403	0.0
1.604	0.0
1.618	0.0
1.695	1.0
1.759	1.0
1.989	0.0
2.117	0.0
2.123	0.0
2.274	0.0
2.289	0.0
2.373	0.0
2.393	0.0
2.597	0.0
3.176	0.0
3.313	0.0
3.379	1.0
3.414	1.0
3.938	0.0
4.129	1.0
4.221	0.0
4.512	0.0
4.821	1.0
5.275	0.0
5.761	0.0
5.780	0.0
5.862	0.0
6.229	0.0
7.003	0.0
7.150	0.0
8.275	1.0
8.569	1.0
8.718	0.0
9.840	0.0
10.350	0.0
10.461	0.0
11.103	0.0
11.120	0.0
13.435	0.0
16.997	1.0
17.645	0.0
******* 29 ******	
0.036	0.0
0.374	0.0
0.676	0.0
0.729	0.0
0.822	0.0
0.882	0.0
0.968	0.0
1.029	0.0
1.370	0.0
1.426	0.0
1.493	0.0
1.590	0.0
1.840	0.0
1.925	0.0
1.927	0.0
2.349	0.0
2.374	0.0
2.414	0.0
2.617	0.0
3.096	0.0
3.140	0.0
3.271	0.0
3.520	1.0
3.764	0.0
3.879	0.0
4.127	1.0
4.448	0.0
4.695	0.0
4.900	0.0
4.919	1.0
5.580	1.0
6.016	1.0
6.166	1.0
6.350	0.0
6.705	0.0
7.086	0.0
7.356	0.0
7.404	1.0
8.018	0.0
8.319	0.0
8.354	0.0
8.413	1.0
8.430	0.0
8.496	0.0
9.478	0.0
9.675	0.0
11.093	1.0
11.554	0.0
12.148	1.0
12.756	0.0

******* 30 *****		7.181	0.0	******* 32 ******	
0.072	0.0	8.441	1.0	0.054	0.0
0.184	0.0	10.889	1.0	0.110	0.0
0.237	0.0	11.004	0.0	0.120	0.0
0.317	0.0	14.059	1.0	0.202	0.0
0.384	0.0	19.538	1.0	0.540	0.0
0.388	1.0	21.683	1.0	0.576	1.0
0.594	0.0	22.406	0.0	0.663	0.0
0.703	1.0	******* 31 ******		0.797	0.0
0.792	0.0	0.252	0.0	0.948	0.0
0.877	0.0	0.622	0.0	0.984	0.0
0.924	0.0	0.665	0.0	1.020	0.0
0.928	0.0	0.693	0.0	1.029	1.0
0.977	0.0	0.749	0.0	1.086	0.0
1.187	1.0	1.092	0.0	1.328	1.0
1.205	0.0	1.095	0.0	1.368	0.0
1.214	0.0	1.098	1.0	1.498	0.0
1.324	0.0	1.142	1.0	1.611	0.0
1.565	0.0	1.163	1.0	1.672	0.0
1.594	1.0	1.823	1.0	2.013	1.0
1.658	0.0	2.015	0.0	2.111	0.0
1.747	0.0	2.314	0.0	2.137	0.0
1.774	0.0	2.486	0.0	2.283	0.0
1.786	0.0	2.669	0.0	2.415	0.0
1.923	0.0	2.754	1.0	2.545	0.0
2.184	0.0	2.978	0.0	2.680	0.0
2.213	1.0	3.047	1.0	2.682	0.0
2.379	0.0	3.258	0.0	2.743	0.0
2.402	0.0	3.552	1.0	2.787	0.0
2.665	0.0	3.629	0.0	3.593	1.0
3.060	0.0	3.980	0.0	3.613	0.0
3.091	0.0	4.089	0.0	4.093	0.0
3.224	0.0	4.569	0.0	4.351	0.0
3.227	0.0	4.983	0.0	4.630	0.0
3.309	0.0	5.009	0.0	4.852	0.0
3.325	0.0	5.242	1.0	4.902	0.0
3.728	0.0	5.660	0.0	6.059	1.0
3.858	0.0	7.722	0.0	7.327	0.0
4.002	0.0	8.026	1.0	7.694	0.0
4.350	0.0	8.126	0.0	7.878	1.0
4.462	0.0	8.496	0.0	8.267	0.0
4.555	0.0	9.938	0.0	10.070	0.0
5.711	0.0	11.521	0.0	11.613	0.0
6.063	0.0	11.952	0.0	11.811	0.0
6.213	0.0	13.524	0.0	12.328	1.0
6.404	0.0	13.964	0.0	12.545	0.0
6.404	0.0	15.157	0.0	15.672	1.0
6.590	0.0	16.659	0.0	16.122	0.0
7.027	0.0	18.288	0.0	20.688	0.0
		24.000	1.0	21.889	1.0

****** 33 *******	
0.128	0.0
0.182	0.0
0.212	0.0
0.276	0.0
0.512	0.0
0.781	0.0
0.873	0.0
1.557	0.0
1.727	0.0
2.042	0.0
2.529	1.0
2.912	0.0
3.224	0.0
3.238	0.0
3.710	0.0
3.770	0.0
3.881	0.0
4.249	0.0
4.569	0.0
5.418	0.0
5.663	0.0
5.860	0.0
6.014	0.0
6.100	0.0
7.291	0.0
7.843	0.0
8.360	1.0
8.490	1.0
8.971	1.0
10.829	0.0
12.722	1.0
13.171	1.0
14.622	1.0
15.152	1.0
18.347	0.0
20.776	1.0
24.000	1.0
****** 34 *******	
0.008	0.0
0.074	0.0
0.108	0.0
0.162	0.0
0.236	0.0
0.261	0.0
0.373	0.0
0.548	0.0
0.617	0.0
0.930	0.0
0.943	0.0
0.958	0.0
0.981	0.0
1.001	0.0
1.031	0.0
1.291	1.0
1.638	0.0
1.743	0.0
1.878	0.0
1.925	0.0
2.060	0.0
2.083	0.0
2.267	0.0
2.276	0.0
3.125	0.0
3.141	0.0
3.240	0.0
3.300	0.0
3.555	0.0
3.867	0.0
3.932	0.0
4.669	0.0
5.013	1.0
5.067	1.0
5.148	0.0
5.563	0.0
5.585	1.0
5.903	1.0
5.917	0.0
6.017	0.0
6.168	1.0
6.185	0.0
6.303	1.0
6.911	1.0
7.400	1.0
7.405	0.0
8.454	0.0
8.475	0.0
8.742	0.0
10.933	0.0
12.495	0.0
14.408	0.0
15.747	0.0
21.940	1.0
******* 35 ******	
0.208	0.0
0.368	1.0
0.636	0.0
0.661	0.0
0.709	0.0
0.906	0.0
1.257	1.0
1.551	0.0
1.936	0.0
2.176	0.0
2.311	0.0
2.312	0.0
2.510	1.0
2.523	0.0
2.666	0.0
2.693	0.0
2.740	0.0
2.970	0.0
3.111	0.0
3.676	1.0
4.369	1.0
5.116	1.0
5.845	0.0
5.902	0.0
6.558	1.0
6.617	1.0
7.889	0.0
8.145	0.0
8.166	0.0
8.960	0.0
10.056	0.0
10.350	0.0
10.469	0.0
11.127	1.0
11.464	0.0
13.105	1.0
13.345	0.0
15.999	0.0
18.273	0.0
20.324	0.0

******* 36 ******	
0.012	0.0
0.212	1.0
0.295	0.0
0.322	0.0
0.566	0.0
0.738	1.0
0.767	1.0
1.077	1.0
1.115	1.0
1.128	0.0
1.267	0.0
1.326	0.0
1.332	0.0
1.431	0.0
1.457	0.0
1.705	0.0
1.757	0.0
1.942	0.0
1.963	0.0
1.964	0.0
2.051	0.0
2.394	1.0
2.404	1.0
2.459	0.0
2.581	1.0
2.893	0.0
2.904	0.0
2.940	0.0
3.394	0.0
3.595	0.0
3.816	0.0
4.039	0.0
4.228	0.0
4.248	0.0
4.397	0.0
4.437	0.0
4.632	0.0
5.118	0.0
5.207	0.0
5.288	0.0
5.816	0.0
6.380	0.0
6.585	1.0
6.842	1.0
7.117	0.0
7.649	0.0
8.759	0.0
11.063	0.0
12.559	0.0
12.767	0.0
13.704	0.0
14.511	0.0
15.346	0.0
15.501	0.0
******* 37 ******	
0.199	0.0
0.265	0.0
0.393	0.0
0.508	1.0
0.515	0.0
0.684	0.0
0.703	0.0
0.741	0.0
1.062	0.0
1.125	0.0
1.157	1.0
1.263	0.0
1.452	1.0
1.671	0.0
1.739	1.0
1.793	0.0
1.843	0.0
1.921	0.0
2.265	0.0
2.387	0.0
2.450	0.0
2.477	0.0
2.658	0.0
2.661	0.0
2.821	0.0
3.003	0.0
3.037	0.0
3.488	0.0
3.590	0.0
3.652	0.0
3.847	0.0
4.028	0.0
4.453	0.0
4.510	1.0
4.511	0.0
4.580	1.0
4.757	0.0
5.271	0.0
5.385	1.0
6.047	0.0
6.143	0.0
6.618	0.0
7.002	0.0
7.056	1.0
7.441	0.0
7.896	0.0
8.061	0.0
8.969	0.0
8.981	1.0
9.637	0.0
10.418	0.0
14.815	0.0
16.559	1.0
19.490	0.0

****** 38 *******		0.897	0.0	******* 40 ******	
0.303	0.0	0.922	0.0	0.023	0.0
0.921	1.0	0.936	0.0	0.084	0.0
0.960	1.0	1.000	0.0	0.294	0.0
0.992	0.0	1.019	0.0	0.477	0.0
1.055	1.0	1.112	1.0	0.610	1.0
1.140	0.0	1.193	0.0	0.667	0.0
1.197	1.0	1.287	0.0	1.037	1.0
1.237	0.0	1.301	0.0	1.325	1.0
1.272	0.0	1.345	0.0	1.512	1.0
1.493	0.0	1.372	0.0	1.596	0.0
1.571	0.0	1.510	0.0	1.770	0.0
1.696	0.0	1.575	1.0	1.901	0.0
2.092	1.0	1.786	0.0	2.111	0.0
2.186	0.0	1.826	0.0	2.271	1.0
2.830	0.0	2.041	0.0	2.437	0.0
2.852	1.0	2.135	0.0	2.659	0.0
3.592	0.0	2.199	0.0	2.953	0.0
5.310	0.0	2.257	0.0	3.252	0.0
5.371	0.0	2.259	0.0	3.404	0.0
5.935	0.0	2.300	0.0	3.425	0.0
6.068	0.0	2.570	0.0	3.500	0.0
6.144	0.0	2.720	0.0	3.658	0.0
6.708	0.0	3.316	0.0	3.687	0.0
7.475	0.0	3.365	0.0	3.726	0.0
7.586	1.0	3.507	0.0	4.066	0.0
7.600	0.0	3.786	0.0	4.471	0.0
8.085	0.0	4.262	0.0	4.736	0.0
8.242	0.0	4.481	1.0	5.268	0.0
8.762	0.0	4.563	1.0	5.754	0.0
9.972	0.0	4.642	0.0	6.174	0.0
10.436	1.0	4.761	0.0	6.670	0.0
10.992	1.0	5.380	0.0	6.868	0.0
11.090	1.0	5.414	0.0	6.910	1.0
11.149	0.0	5.546	0.0	7.177	1.0
11.515	0.0	5.836	0.0	8.677	0.0
11.625	0.0	6.607	1.0	9.492	0.0
13.629	0.0	7.077	0.0	9.613	1.0
17.669	0.0	7.836	1.0	10.586	0.0
21.249	0.0	8.648	0.0	11.143	0.0
******* 39 ******		8.713	0.0	12.087	0.0
0.110	0.0	9.141	0.0	12.467	1.0
0.163	0.0	9.358	0.0	14.411	0.0
0.339	1.0	9.683	1.0	21.049	0.0
0.344	1.0	9.765	1.0	24.000	1.0
0.437	0.0	9.884	0.0		
0.496	0.0	10.352	0.0		
0.602	0.0	10.644	0.0		
0.689	0.0	12.465	0.0		
0.811	0.0	19.418	0.0		

****** 41 *******

0.048	0.0
0.089	1.0
0.193	1.0
0.567	0.0
0.901	0.0
1.156	0.0
1.198	1.0
1.282	0.0
1.433	0.0
1.560	0.0
1.596	0.0
1.724	0.0
1.942	1.0
1.994	0.0
2.075	0.0
2.236	0.0
2.327	0.0
2.439	0.0
2.475	0.0
2.691	0.0
3.359	0.0
3.368	1.0
3.475	0.0
3.608	1.0
3.734	1.0
3.920	0.0
4.210	0.0
4.279	0.0
4.357	1.0
4.485	0.0
4.543	1.0
4.682	0.0
5.429	0.0
5.815	0.0
6.108	0.0
6.191	0.0
6.944	0.0
7.269	0.0
7.520	0.0
9.027	0.0
9.154	0.0
9.633	0.0
9.759	0.0
12.214	0.0
15.033	0.0
15.994	1.0
17.941	0.0
18.024	0.0

******* 42 ******

0.270	0.0
0.362	1.0
0.886	0.0
0.949	0.0
0.979	0.0
1.299	0.0
1.343	0.0
1.349	0.0
1.455	1.0
1.485	0.0
1.613	0.0
1.959	0.0
2.404	0.0
2.470	0.0
3.274	0.0
3.306	0.0
3.377	0.0
3.516	0.0
3.776	0.0
3.796	0.0
3.998	0.0
4.667	1.0
4.893	0.0
6.265	0.0
6.545	1.0
7.624	0.0
7.798	0.0
7.870	0.0
8.175	0.0
9.450	0.0
9.469	1.0
10.288	1.0
10.975	0.0
13.025	1.0
14.155	0.0
15.162	0.0
15.490	1.0
20.280	1.0
24.000	1.0

******* 43 ******

0.147	0.0
0.419	0.0
0.632	0.0
0.731	0.0
0.881	1.0
0.929	0.0
1.218	1.0
1.260	0.0
1.269	0.0
1.666	0.0
1.892	0.0
2.148	0.0
2.225	0.0
2.300	0.0
3.051	1.0
3.081	1.0
3.220	0.0
3.340	0.0
3.447	0.0
3.541	0.0
3.751	0.0
3.809	0.0
3.894	0.0
4.218	0.0
4.264	1.0
4.484	1.0
5.037	1.0
5.191	0.0
5.329	0.0
5.333	0.0
5.635	0.0
5.820	0.0
5.837	0.0
6.025	0.0
6.476	0.0
6.625	0.0
6.871	0.0
7.143	0.0
8.155	1.0
8.384	0.0
8.626	0.0
8.891	0.0
9.115	1.0
9.879	0.0
11.104	0.0
11.568	1.0
13.409	0.0
17.730	0.0

******* 44 ******		****** 45 *******		8.049	0.0
0.135	0.0	0.075	0.0	8.137	0.0
0.173	0.0	0.258	0.0	8.188	0.0
0.237	0.0	0.311	1.0	8.954	0.0
0.239	0.0	0.415	1.0	9.128	0.0
0.337	0.0	0.429	1.0	9.303	0.0
0.357	0.0	0.553	0.0	10.702	0.0
0.372	0.0	0.557	0.0	11.770	0.0
0.844	0.0	0.789	0.0	13.277	0.0
1.299	1.0	0.921	1.0	13.652	0.0
1.333	0.0	0.937	0.0		
1.435	0.0	0.985	0.0		
1.553	0.0	0.991	0.0		
1.664	0.0	0.991	0.0		
1.700	0.0	1.011	0.0		
2.179	1.0	1.164	0.0		
2.229	0.0	1.168	0.0		
2.238	0.0	1.289	0.0		
2.264	0.0	1.350	0.0		
2.316	0.0	1.405	1.0		
2.351	0.0	1.427	0.0		
2.371	0.0	1.489	0.0		
2.514	0.0	1.520	0.0		
2.689	0.0	1.982	1.0		
2.887	1.0	2.184	0.0		
2.939	1.0	2.341	1.0		
3.158	0.0	2.522	0.0		
3.209	0.0	2.549	0.0		
3.265	0.0	2.592	0.0		
3.426	0.0	2.601	0.0		
3.672	0.0	3.300	0.0		
3.769	1.0	3.475	0.0		
3.778	0.0	3.722	1.0		
4.152	0.0	4.105	0.0		
4.326	1.0	4.184	0.0		
4.687	0.0	4.344	1.0		
6.488	1.0	4.510	0.0		
6.684	0.0	4.664	0.0		
7.289	1.0	4.717	0.0		
7.522	0.0	4.875	0.0		
8.362	0.0	5.021	1.0		
8.798	0.0	5.150	0.0		
9.661	0.0	5.317	0.0		
9.848	1.0	5.667	0.0		
9.966	0.0	5.718	0.0		
12.089	1.0	5.887	0.0		
12.143	0.0	6.339	0.0		
12.783	0.0	6.885	0.0		
14.086	0.0	6.932	0.0		
18.796	0.0	7.223	0.0		
19.387	0.0				

******* 46 ******	
0.002	0.0
0.177	0.0
0.299	0.0
0.514	0.0
0.846	0.0
0.901	0.0
0.970	0.0
1.003	0.0
1.192	0.0
1.270	0.0
1.387	0.0
1.397	0.0
1.515	1.0
1.818	0.0
1.976	0.0
2.131	0.0
2.153	0.0
2.238	0.0
2.493	1.0
2.803	0.0
2.886	1.0
2.980	0.0
2.984	0.0
3.128	0.0
3.296	1.0
3.380	0.0
3.411	0.0
3.646	0.0
3.732	1.0
4.077	0.0
4.242	1.0
4.385	0.0
4.456	0.0
4.618	0.0
4.788	0.0
5.081	0.0
5.678	0.0
6.801	0.0
6.904	0.0
6.962	0.0
7.144	0.0
7.382	0.0
7.597	0.0
8.761	0.0
8.815	1.0
8.826	0.0
8.852	1.0
9.601	0.0
10.095	1.0
10.494	0.0
15.185	0.0
22.730	1.0
****** 47 *******	
0.056	0.0
0.146	0.0
0.185	0.0
0.231	0.0
0.303	1.0
0.329	0.0
0.361	0.0
0.686	0.0
0.835	0.0
0.869	0.0
1.069	1.0
1.086	0.0
1.097	0.0
1.351	0.0
1.379	0.0
1.406	0.0
1.484	0.0
1.518	0.0
1.621	0.0
1.701	1.0
1.744	0.0
1.749	1.0
1.756	0.0
1.797	0.0
1.902	0.0
1.972	0.0
1.982	0.0
2.016	1.0
2.268	0.0
2.299	0.0
2.378	0.0
2.483	0.0
2.519	0.0
2.617	1.0
2.706	0.0
2.983	0.0
3.156	0.0
3.541	0.0
4.037	0.0
4.059	0.0
4.155	0.0
4.701	0.0
5.216	0.0
5.467	0.0
5.711	0.0
6.123	0.0
6.265	0.0
6.321	0.0
6.758	0.0
7.232	1.0
9.661	1.0
9.727	0.0
10.280	0.0
10.342	0.0
11.965	0.0
13.009	0.0
14.265	1.0
17.160	1.0
17.966	0.0

******* 48 ******

0.066	0.0
0.211	0.0
0.255	0.0
0.315	0.0
0.422	0.0
0.429	0.0
0.610	0.0
0.628	1.0
0.661	0.0
0.724	1.0
0.827	0.0
0.901	0.0
0.934	0.0
1.023	0.0
1.139	0.0
1.216	0.0
1.256	1.0
1.326	0.0
1.579	1.0
1.915	0.0
2.317	0.0
2.317	0.0
2.433	0.0
2.593	0.0
2.734	0.0
2.944	0.0
2.990	0.0
3.519	0.0
3.692	1.0
3.849	1.0
3.941	0.0
4.052	0.0
4.160	0.0
4.825	0.0
4.835	1.0
4.868	1.0
5.040	0.0
5.122	0.0
5.480	0.0
6.129	0.0
6.697	0.0
6.712	0.0
6.813	0.0
7.053	0.0
7.177	1.0
7.466	1.0
8.178	0.0
8.273	0.0
9.110	0.0
9.940	0.0
10.872	0.0
11.227	0.0
11.675	0.0
16.637	0.0
17.895	0.0

******** 49 *****

0.233	0.0
0.260	0.0
0.324	0.0
0.357	0.0
0.403	0.0
0.434	0.0
0.580	0.0
1.136	0.0
1.557	1.0
1.822	1.0
3.015	1.0
3.695	0.0
4.432	0.0
4.519	0.0
4.998	0.0
5.035	0.0
5.115	0.0
5.429	0.0
5.743	1.0
5.818	0.0
5.838	0.0
5.981	1.0
8.329	0.0
8.895	0.0
11.332	0.0
12.989	0.0
17.924	0.0
18.162	1.0
23.643	1.0
24.000	1.0
24.000	1.0
24.000	1.0

******* 50 ******

0.062	0.0
0.117	0.0
0.252	1.0
0.309	0.0
0.650	0.0
0.692	0.0
0.924	1.0
1.087	0.0
1.160	0.0
1.403	0.0
1.799	1.0
2.002	0.0
2.209	1.0
2.544	0.0
2.623	0.0
2.834	0.0
2.921	0.0
2.947	0.0
2.982	1.0
3.234	0.0
3.324	0.0
3.575	0.0
3.860	0.0
3.888	0.0
4.061	0.0
4.256	0.0
4.453	1.0
4.556	1.0
5.306	0.0
5.417	1.0
5.463	0.0
5.640	0.0
5.641	0.0
5.644	1.0
5.710	1.0
6.745	0.0
7.833	0.0
8.169	0.0
9.109	0.0
9.505	0.0
9.535	0.0
9.945	0.0
10.749	0.0
11.355	0.0
11.407	0.0
18.356	0.0
23.748	0.0

******** 51 *****	
0.010	0.0
0.032	0.0
0.183	0.0
0.452	0.0
0.458	1.0
0.516	0.0
0.548	0.0
0.598	0.0
0.755	0.0
0.791	0.0
0.861	1.0
1.064	0.0
1.094	0.0
1.360	0.0
1.540	0.0
1.581	1.0
1.880	0.0
1.884	0.0
2.009	0.0
2.045	0.0
2.297	0.0
2.524	0.0
2.627	0.0
2.690	0.0
2.864	0.0
2.900	0.0
3.041	1.0
3.101	0.0
3.336	0.0
3.486	0.0
3.600	1.0
3.845	0.0
4.132	0.0
4.196	1.0
4.292	0.0
4.367	0.0
4.450	0.0
4.471	0.0
4.476	0.0
4.500	0.0
4.522	1.0
4.727	1.0
4.913	0.0
4.945	0.0
5.301	0.0
5.310	0.0
5.639	0.0
6.221	1.0
6.876	1.0
7.374	0.0
8.250	0.0
8.332	0.0
8.727	0.0
8.874	0.0
9.716	0.0
9.784	0.0
11.168	0.0
12.672	0.0
15.792	0.0
****** 52 *******	
0.074	0.0
0.116	0.0
0.237	0.0
0.629	0.0
0.778	0.0
0.845	1.0
1.308	0.0
1.809	0.0
2.011	1.0
2.167	1.0
2.543	0.0
3.160	0.0
3.165	0.0
3.269	1.0
3.495	0.0
3.548	0.0
3.702	0.0
3.768	0.0
3.841	1.0
5.411	0.0
5.886	0.0
6.550	1.0
6.735	0.0
7.896	0.0
8.030	0.0
8.091	0.0
8.786	0.0
9.128	0.0
9.369	1.0
9.515	1.0
9.552	0.0
10.231	1.0
10.530	0.0
11.088	0.0
12.701	0.0
14.049	0.0
21.989	0.0
24.000	1.0
******* 53 ******	
0.213	0.0
0.365	0.0
0.396	0.0
0.420	0.0
0.750	0.0
0.999	0.0
1.265	0.0
1.342	0.0
1.347	0.0
1.364	0.0
1.370	1.0
1.392	0.0
1.408	0.0
1.634	0.0
2.109	0.0
2.175	0.0
2.175	1.0
2.410	1.0
2.632	0.0
2.793	0.0
2.940	1.0
3.085	1.0
3.328	0.0
3.344	0.0
3.648	0.0
3.675	0.0
3.817	0.0
3.925	0.0
3.974	0.0
4.815	1.0
4.900	0.0
4.923	0.0
5.884	0.0
5.884	0.0
5.916	0.0
6.017	0.0
6.822	1.0
7.660	0.0
8.244	0.0
9.141	0.0
10.451	0.0
10.783	0.0
11.975	0.0
12.025	1.0
12.096	0.0
12.258	1.0
15.909	0.0
24.000	1.0

******* 54 ******

0.118	0.0
0.175	0.0
0.190	0.0
0.825	0.0
0.958	0.0
0.974	0.0
1.025	0.0
1.361	0.0
1.478	0.0
1.712	1.0
1.737	1.0
2.010	0.0
2.150	0.0
2.716	0.0
2.843	0.0
2.915	0.0
2.933	1.0
2.945	0.0
2.979	0.0
3.013	0.0
3.150	0.0
3.246	0.0
3.291	1.0
3.432	0.0
3.553	0.0
3.812	1.0
4.036	0.0
4.058	1.0
4.091	0.0
4.423	0.0
4.469	0.0
4.555	0.0
4.598	0.0
4.759	0.0
4.909	0.0
4.926	0.0
4.929	0.0
5.134	0.0
5.467	0.0
5.514	0.0
5.562	0.0
5.629	1.0
5.749	0.0
6.487	0.0
6.879	0.0
7.144	0.0
7.478	0.0
9.182	1.0
10.018	0.0
10.037	0.0
10.087	0.0
10.518	0.0
11.591	1.0
12.227	1.0

****** 55 *******

0.012	0.0
0.700	0.0
0.769	0.0
0.838	0.0
1.139	0.0
1.277	1.0
1.450	0.0
1.660	0.0
1.709	0.0
1.891	1.0
2.129	0.0
2.276	1.0
2.772	0.0
2.880	1.0
4.393	0.0
4.786	1.0
5.051	0.0
5.387	0.0
5.476	0.0
5.486	0.0
5.727	0.0
6.580	0.0
6.826	0.0
7.330	0.0
7.378	0.0
7.793	0.0
8.034	0.0
8.087	1.0
8.120	0.0
8.223	0.0
9.003	0.0
9.052	0.0
10.243	0.0
11.699	1.0
14.827	1.0
16.622	1.0
18.376	0.0
24.000	1.0

******* 56 ******

0.362	0.0
0.422	1.0
0.423	1.0
0.564	0.0
0.641	0.0
0.706	0.0
0.725	0.0
0.827	1.0
1.029	0.0
1.162	1.0
1.179	0.0
1.501	0.0
1.559	0.0
1.613	1.0
1.691	0.0
1.781	1.0
1.937	0.0
2.201	0.0
2.337	0.0
2.459	0.0
2.462	0.0
2.491	0.0
2.828	0.0
3.164	0.0
3.224	0.0
3.298	0.0
3.483	0.0
3.588	0.0
3.697	0.0
4.060	0.0
4.166	0.0
4.831	0.0
4.914	1.0
5.493	0.0
5.534	0.0
5.607	1.0
5.767	0.0
5.899	0.0
6.050	0.0
7.896	0.0
9.009	0.0
9.182	1.0
9.337	0.0
9.443	0.0
10.714	0.0
11.860	0.0
11.900	0.0
12.944	0.0
18.043	0.0
24.000	1.0

******* 57 ******	
0.137	0.0
0.212	0.0
0.253	0.0
0.460	0.0
0.504	0.0
0.709	1.0
0.712	1.0
0.770	0.0
0.874	0.0
1.185	0.0
1.544	0.0
1.552	0.0
1.982	1.0
2.035	0.0
2.052	0.0
2.059	0.0
2.140	0.0
2.462	0.0
2.646	1.0
2.648	0.0
2.815	0.0
3.031	0.0
3.110	1.0
3.280	0.0
3.401	0.0
3.435	0.0
4.303	0.0
5.119	0.0
5.163	0.0
7.053	0.0
7.239	0.0
7.427	0.0
7.918	0.0
7.968	0.0
8.095	1.0
8.253	0.0
8.357	0.0
8.409	1.0
8.610	0.0
9.095	0.0
10.116	1.0
10.617	0.0
11.146	1.0
12.875	0.0
15.067	0.0
15.303	1.0
15.860	0.0
******** 58 *****	
0.052	0.0
0.390	0.0
0.731	0.0
0.757	1.0
0.766	0.0
1.092	1.0
1.103	0.0
1.368	1.0
1.418	0.0
1.428	0.0
1.817	0.0
2.171	0.0
2.334	0.0
2.462	0.0
2.559	0.0
2.637	0.0
3.005	0.0
3.053	0.0
3.291	0.0
3.309	0.0
3.394	1.0
3.648	0.0
3.717	1.0
3.912	0.0
4.103	1.0
4.525	0.0
4.839	0.0
4.915	0.0
5.925	0.0
6.244	0.0
6.502	1.0
6.561	0.0
7.262	0.0
7.345	0.0
7.826	0.0
8.171	0.0
9.971	0.0
10.000	1.0
10.839	0.0
12.661	0.0
13.369	0.0
13.789	1.0
21.829	1.0
22.908	0.0
******* 59 ******	
0.177	0.0
0.230	1.0
0.289	0.0
0.338	0.0
0.431	0.0
0.477	0.0
0.565	1.0
0.721	0.0
0.817	0.0
0.819	0.0
0.885	0.0
1.346	0.0
1.552	0.0
1.626	0.0
1.654	0.0
1.707	0.0
1.730	0.0
1.853	1.0
1.856	0.0
1.953	1.0
1.977	1.0
2.126	1.0
2.158	0.0
2.211	1.0
2.429	1.0
2.623	0.0
2.638	0.0
2.857	0.0
2.927	0.0
3.169	0.0
3.392	0.0
3.497	0.0
3.714	0.0
3.795	0.0
3.797	0.0
3.830	0.0
4.422	0.0
4.427	0.0
4.449	0.0
4.567	0.0
4.870	0.0
5.055	0.0
5.427	1.0
6.128	0.0
6.596	0.0
6.984	0.0
7.181	0.0
7.327	0.0
8.078	0.0
8.794	0.0
9.592	0.0
9.940	0.0
10.495	1.0
11.882	0.0
13.889	0.0
14.559	0.0
17.174	0.0

******** 60 *****		17.835	1.0	14.486	0.0
0.098	0.0	24.000	1.0	24.000	1.0
0.110	0.0	******* 61 ******			
0.172	0.0	0.137	0.0		
0.182	0.0	0.232	0.0		
0.183	0.0	0.235	0.0		
0.417	0.0	0.510	0.0		
0.543	0.0	0.640	1.0		
0.625	0.0	0.790	1.0		
0.790	0.0	0.802	0.0		
0.814	1.0	0.869	0.0		
1.016	0.0	1.021	0.0		
1.100	0.0	1.077	0.0		
1.147	0.0	1.151	0.0		
1.182	0.0	1.156	0.0		
1.299	1.0	1.367	0.0		
1.875	0.0	1.405	0.0		
1.964	0.0	2.092	0.0		
2.185	0.0	2.441	0.0		
2.289	0.0	2.674	0.0		
2.440	0.0	2.713	1.0		
2.577	0.0	2.740	0.0		
2.646	0.0	2.753	1.0		
2.680	0.0	2.859	1.0		
2.736	0.0	3.230	0.0		
2.917	0.0	3.459	1.0		
3.263	0.0	3.979	0.0		
3.449	0.0	4.011	0.0		
3.472	0.0	4.344	0.0		
3.745	0.0	4.344	0.0		
3.795	0.0	4.665	0.0		
3.906	0.0	4.704	0.0		
4.471	0.0	4.708	0.0		
4.755	0.0	4.811	0.0		
6.241	1.0	5.678	0.0		
6.277	1.0	5.852	0.0		
6.332	0.0	6.201	0.0		
6.497	0.0	6.384	0.0		
7.396	1.0	6.655	0.0		
7.564	0.0	7.242	0.0		
7.613	0.0	7.535	1.0		
8.154	1.0	7.668	0.0		
8.647	0.0	8.539	0.0		
8.660	0.0	10.296	0.0		
9.132	0.0	10.520	1.0		
9.874	0.0	10.577	1.0		
9.973	1.0	11.140	0.0		
15.340	1.0	12.354	0.0		
15.621	0.0	12.951	0.0		

******* 62 ******	
0.033	0.0
0.129	0.0
0.346	0.0
0.427	1.0
0.509	0.0
0.510	1.0
0.519	1.0
0.545	0.0
0.552	0.0
0.684	0.0
0.728	0.0
0.755	0.0
0.791	1.0
0.902	0.0
0.913	0.0
0.983	0.0
1.059	0.0
1.406	0.0
1.461	0.0
1.517	0.0
1.561	0.0
1.652	0.0
1.744	0.0
1.767	0.0
1.802	0.0
1.887	0.0
1.914	0.0
2.094	0.0
2.215	0.0
2.335	0.0
2.378	0.0
2.418	1.0
2.453	0.0
2.546	0.0
2.771	0.0
2.821	1.0
3.055	0.0
3.839	0.0
3.925	0.0
4.366	0.0
4.651	0.0
4.780	1.0
4.942	0.0
4.982	0.0
4.984	0.0
4.993	0.0
5.221	0.0
5.558	0.0
5.780	0.0
6.007	0.0
6.353	0.0
6.369	0.0
6.595	0.0
6.845	1.0
7.150	0.0
7.733	0.0
9.741	0.0
10.355	0.0
14.259	1.0
16.059	0.0
16.086	1.0
16.248	0.0
******* 63 ******	
0.039	0.0
0.095	0.0
0.135	0.0
0.169	1.0
0.200	0.0
0.282	1.0
1.054	0.0
1.068	0.0
1.160	0.0
1.298	0.0
1.484	0.0
1.522	0.0
1.602	0.0
1.686	0.0
2.052	0.0
2.106	0.0
2.289	1.0
2.698	0.0
3.109	1.0
3.363	0.0
3.926	0.0
3.975	0.0
4.149	0.0
4.856	0.0
5.292	0.0
5.399	0.0
5.883	1.0
5.890	0.0
6.532	0.0
6.655	0.0
7.128	0.0
7.682	0.0
7.868	1.0
8.484	0.0
8.633	0.0
8.718	1.0
8.945	0.0
10.401	1.0
10.435	0.0
12.037	0.0
13.596	0.0
14.513	1.0
19.979	1.0
21.612	0.0

****** 64 *******	
0.053	0.0
0.321	0.0
0.369	0.0
0.528	0.0
0.647	0.0
0.725	0.0
0.973	0.0
0.990	1.0
1.035	0.0
1.045	0.0
1.144	1.0
1.244	0.0
1.269	0.0
1.358	0.0
1.398	0.0
1.416	0.0
1.443	1.0
1.755	0.0
1.791	0.0
1.832	1.0
1.951	0.0
1.981	0.0
2.209	0.0
2.243	0.0
2.370	0.0
2.602	0.0
2.721	1.0
2.801	0.0
2.817	0.0
2.832	0.0
2.859	1.0
3.086	0.0
3.177	0.0
3.441	0.0
4.114	0.0
4.120	0.0
4.610	0.0
4.764	0.0
5.941	1.0
6.298	0.0
7.219	0.0
7.903	0.0
8.006	0.0
8.838	0.0
9.250	0.0
10.038	0.0
10.794	0.0
12.248	0.0
13.764	1.0
14.882	0.0
14.921	0.0
14.922	1.0
18.943	1.0
****** 65 *******	
0.057	0.0
0.372	0.0
0.462	1.0
0.524	0.0
0.549	0.0
0.576	0.0
0.591	1.0
0.846	0.0
0.907	0.0
1.287	0.0
1.447	0.0
1.508	0.0
1.624	0.0
1.638	0.0
1.972	1.0
2.002	0.0
2.100	0.0
2.309	1.0
2.462	0.0
2.812	0.0
3.139	0.0
3.636	0.0
4.227	1.0
4.231	1.0
4.503	0.0
4.624	0.0
4.626	0.0
4.807	0.0
4.811	0.0
5.155	0.0
5.265	0.0
5.454	0.0
5.464	1.0
5.722	0.0
5.974	0.0
6.562	0.0
6.569	1.0
7.518	0.0
7.637	0.0
7.742	0.0
7.792	0.0
8.134	0.0
8.476	0.0
9.319	0.0
9.860	0.0
10.393	0.0
10.823	0.0
11.334	1.0
13.134	1.0
17.028	0.0

******* 66 ******	
0.213	0.0
0.222	1.0
0.250	0.0
0.251	1.0
0.392	0.0
0.450	0.0
0.491	0.0
0.586	0.0
0.600	0.0
0.641	0.0
0.684	0.0
0.750	0.0
0.836	0.0
0.928	0.0
0.979	0.0
0.995	1.0
1.176	0.0
1.220	0.0
1.468	0.0
1.916	0.0
2.027	0.0
2.095	0.0
2.148	0.0
2.298	1.0
2.309	0.0
2.529	0.0
2.782	1.0
2.818	0.0
2.876	0.0
3.125	0.0
3.228	0.0
3.236	1.0
3.286	0.0
3.645	0.0
3.885	0.0
4.629	0.0
5.131	0.0
5.133	0.0
5.223	0.0
5.560	0.0
5.665	0.0
5.738	0.0
6.117	0.0
6.300	0.0
6.468	0.0
6.711	0.0
7.169	0.0
7.476	1.0
8.349	0.0
8.377	1.0
9.078	0.0
9.247	1.0
9.251	0.0
9.863	0.0
10.207	0.0
12.337	0.0
12.528	0.0
16.106	1.0
******** 67 *****	
0.189	0.0
0.413	1.0
0.461	0.0
0.545	0.0
0.582	0.0
0.810	0.0
1.060	1.0
1.094	0.0
1.098	0.0
1.771	1.0
2.283	1.0
3.044	0.0
3.267	0.0
3.279	0.0
3.437	0.0
3.611	0.0
3.663	0.0
3.958	1.0
4.233	0.0
4.507	0.0
4.588	1.0
5.536	0.0
5.540	0.0
5.622	0.0
5.835	0.0
6.071	0.0
6.145	1.0
6.284	0.0
6.613	1.0
8.521	0.0
8.805	0.0
8.957	0.0
9.328	0.0
9.707	0.0
9.931	1.0
10.127	0.0
12.020	0.0
13.272	0.0
13.464	0.0
18.519	0.0
21.808	1.0

******* 68 ******	
0.133	0.0
0.266	0.0
0.336	0.0
0.339	0.0
0.363	0.0
0.374	0.0
0.520	0.0
0.618	0.0
0.726	0.0
0.854	0.0
0.895	0.0
0.941	0.0
0.975	1.0
1.008	0.0
1.084	0.0
1.355	1.0
1.652	0.0
1.829	0.0
2.065	0.0
2.119	1.0
2.169	0.0
2.233	0.0
2.298	0.0
2.493	0.0
2.638	0.0
2.984	0.0
3.062	1.0
3.500	0.0
3.541	1.0
3.744	0.0
4.464	1.0
4.709	0.0
4.731	0.0
4.868	0.0
4.881	0.0
4.935	0.0
5.093	0.0
5.970	0.0
6.137	0.0
6.250	1.0
6.798	0.0
7.313	0.0
7.396	0.0
7.428	0.0
8.575	0.0
9.114	0.0
11.275	0.0
11.280	0.0
12.377	1.0
13.811	1.0
14.767	0.0
14.853	1.0
15.859	0.0
******* 69 ******	
0.120	0.0
0.273	0.0
0.410	1.0
0.650	0.0
0.757	0.0
0.807	0.0
0.907	0.0
0.915	1.0
0.916	0.0
1.186	0.0
1.304	0.0
1.565	0.0
1.624	0.0
1.749	0.0
1.875	0.0
1.985	0.0
2.281	0.0
2.363	1.0
2.391	0.0
3.135	0.0
3.153	1.0
3.487	0.0
3.516	0.0
4.377	0.0
4.539	0.0
4.630	0.0
4.653	0.0
4.787	0.0
5.015	1.0
5.065	0.0
5.705	0.0
6.360	1.0
8.087	0.0
8.680	1.0
8.707	0.0
9.252	1.0
9.278	0.0
9.496	0.0
9.882	0.0
10.095	0.0
11.126	1.0
11.587	1.0
12.413	0.0
15.320	0.0
15.340	0.0
18.236	0.0

******** 70 *****	
0.004	0.0
0.341	0.0
0.427	1.0
0.532	0.0
0.715	0.0
0.806	0.0
0.817	1.0
1.109	0.0
1.207	1.0
1.210	0.0
1.238	0.0
1.240	0.0
1.392	0.0
1.491	1.0
1.784	0.0
1.941	0.0
2.123	0.0
2.128	0.0
2.202	0.0
3.002	0.0
3.251	1.0
3.257	0.0
3.274	0.0
3.445	0.0
3.640	0.0
3.847	0.0
3.942	0.0
3.972	0.0
4.329	0.0
4.416	0.0
5.033	1.0
5.469	0.0
5.764	0.0
6.346	0.0
6.580	0.0
6.886	0.0
7.425	0.0
7.738	0.0
8.140	1.0
8.340	0.0
8.611	0.0
9.970	0.0
10.450	0.0
10.502	1.0
14.149	1.0
15.660	1.0
19.909	0.0
19.944	0.0

******* 71 ******	
0.199	0.0
0.212	0.0
0.233	0.0
0.324	0.0
0.364	0.0
0.434	0.0
0.627	0.0
0.740	0.0
0.804	1.0
0.941	0.0
1.024	0.0
1.132	0.0
1.137	0.0
1.374	1.0
1.391	0.0
1.552	0.0
1.987	0.0
2.002	0.0
2.272	1.0
2.410	0.0
2.724	0.0
2.825	0.0
3.332	0.0
3.358	0.0
3.434	0.0
3.552	0.0
3.742	0.0
4.588	1.0
4.592	0.0
4.742	0.0
4.858	0.0
5.009	0.0
5.026	0.0
5.380	0.0
5.434	1.0
5.609	0.0
5.630	1.0
5.979	0.0
6.059	1.0
6.416	0.0
7.074	0.0
7.428	1.0
7.466	0.0
7.522	0.0
8.869	0.0
9.256	0.0
9.697	0.0
12.015	0.0
15.059	1.0
18.168	0.0
24.000	1.0

******* 72 ******	
0.033	0.0
0.034	0.0
0.214	0.0
0.358	0.0
0.598	1.0
1.019	0.0
1.652	0.0
1.685	0.0
1.898	1.0
1.912	0.0
1.921	1.0
2.031	0.0
2.097	0.0
2.242	0.0
2.342	0.0
2.486	0.0
2.671	0.0
2.803	0.0
2.953	0.0
3.087	0.0
3.125	0.0
3.297	0.0
3.411	1.0
3.718	0.0
3.798	0.0
3.912	1.0
3.932	0.0
4.056	1.0
4.088	0.0
4.408	0.0
4.421	0.0
4.642	0.0
4.970	0.0
5.906	0.0
7.314	1.0
7.375	0.0
7.426	0.0
7.432	0.0
7.744	0.0
8.520	0.0
10.386	0.0
10.595	0.0
12.648	1.0
13.521	0.0
13.614	1.0
18.340	1.0
23.368	0.0

****** 73 *******	
0.249	0.0
0.285	0.0
0.303	1.0
0.431	0.0
0.480	0.0
0.773	0.0
0.843	0.0
1.134	0.0
1.212	0.0
1.514	0.0
1.600	1.0
1.629	0.0
1.796	0.0
1.876	0.0
2.010	1.0
2.184	0.0
2.280	0.0
2.749	0.0
2.777	1.0
2.909	0.0
2.941	0.0
3.050	1.0
3.063	0.0
3.176	0.0
3.415	1.0
3.457	0.0
3.787	0.0
3.874	0.0
4.423	0.0
4.515	0.0
5.651	0.0
5.858	1.0
6.215	1.0
6.327	0.0
6.376	0.0
6.471	0.0
7.543	0.0
10.441	1.0
10.472	0.0
11.945	0.0
13.350	0.0
13.604	0.0
14.714	0.0
16.266	0.0
16.461	0.0
23.569	1.0
********* 74 *****	
0.111	0.0
0.152	0.0
0.187	0.0
0.206	0.0
0.213	0.0
0.283	0.0
0.286	0.0
0.342	0.0
0.445	0.0
0.449	0.0
0.490	0.0
0.617	1.0
0.861	0.0
0.899	0.0
0.929	0.0
0.943	0.0
0.987	0.0
1.098	0.0
1.246	0.0
1.343	0.0
1.363	0.0
1.515	0.0
1.520	0.0
1.525	1.0
1.640	1.0
1.813	0.0
2.066	0.0
2.150	0.0
2.234	0.0
2.293	1.0
2.294	0.0
2.419	1.0
2.592	0.0
2.673	0.0
2.676	0.0
2.731	0.0
2.873	0.0
2.879	0.0
2.894	0.0
3.005	0.0
3.041	0.0
3.065	1.0
3.182	0.0
3.246	0.0
3.460	0.0
3.476	1.0
3.628	0.0
3.821	0.0
4.241	0.0
4.412	0.0
4.536	1.0
4.779	0.0
5.129	0.0
5.419	1.0
5.675	0.0
6.058	0.0
6.140	0.0
6.144	0.0
6.658	0.0
7.819	0.0
9.407	1.0
9.414	0.0
9.524	0.0
10.011	0.0
10.539	0.0
11.493	0.0
13.970	0.0
14.475	0.0

****** 75*******

0.026	0.0
0.167	0.0
0.218	0.0
0.277	0.0
0.291	0.0
0.306	0.0
0.426	0.0
0.461	0.0
0.588	0.0
0.670	0.0
0.719	0.0
0.819	0.0
0.845	1.0
1.069	0.0
1.158	0.0
1.417	0.0
1.512	0.0
1.799	0.0
2.032	0.0
2.137	0.0
2.351	1.0
2.577	1.0
3.040	0.0
3.117	0.0
3.194	0.0
3.224	1.0
3.514	1.0
3.571	0.0
3.827	0.0
3.881	0.0
3.934	0.0
4.029	0.0
4.120	1.0
4.447	0.0
4.790	0.0
4.941	0.0
5.278	0.0
5.837	0.0
6.005	0.0
6.851	0.0
7.036	0.0
7.416	0.0
7.446	1.0
7.822	0.0
8.791	0.0
8.884	1.0
9.108	0.0
9.235	0.0
10.415	0.0
10.856	0.0
12.566	0.0
16.964	1.0
24.000	1.0

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